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1.Laplace Transformation in Control System

Solved Examples

Find the Laplace transform of the following functions.

- Constant:

$$f(t) = A, \quad t \rangle 0$$

$$L\{A\} = \frac{A}{p}$$

$$1.\ L\{A\} = \int_0^\infty A \cdot e^{-pt} dt = A\left[-\frac{1}{p} \cdot e^{-pt}\right]_0^\infty = A\left(-\frac{1}{p} \cdot 0 + \frac{1}{p} \cdot 1\right) = \frac{A}{p}$$

- Exponencial function:

$$f(t) = e^{-at}, \quad a \rangle 0$$

$$L\{e^{-at}\} = \frac{1}{p+a}$$

$$2.\ L\{e^{-at}\} = \int_0^\infty e^{-at} \cdot e^{-pt}\, dt = \int_0^\infty e^{-(a+p)t} dt = \left[-\frac{1}{a+p} e^{-(a+p)t}\right]_0^\infty = 0 + \frac{1}{p+a}$$

- Sinusoidal functions:

$$\sin \omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \qquad \cos \omega t = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

$$3.\ L\{\sin \omega t\} = \frac{1}{2j} \int_0^\infty \left(e^{j\omega t} - e^{-j\omega t}\right) \cdot e^{-pt} dt = \frac{1}{2j}\left[\int_0^\infty \left(e^{-(p-j\omega)t} dt - \int_0^\infty e^{-(p+j\omega)t} dt \right)\right] =$$

$$\frac{1}{2j}\left(\frac{1}{p-j\omega} - \frac{1}{p+j\omega}\right) = \frac{\omega}{p^2+\omega^2}$$

$$f(t) = \sin(\omega t) \quad \Rightarrow \quad F(p) = \frac{\omega}{p^2+\omega^2}$$

$$f(t) = \cos(\omega t) \quad \Rightarrow \quad F(p) = \frac{p}{p^2+\omega^2}$$

Final value theorem - examples

- Find $f(\infty)$ if $F(p)$ is given by:

$$F(p)=\frac{1}{p-5}$$

$$F(p)=\frac{p-2}{p(p+1)}$$

$$F(p)=\frac{p+5}{p(p+2-j)(p+2+j)}$$

$$F(p)=\frac{1}{(p+j)(p-j)}$$

$$F(p)=\frac{p+10}{(p+2)(p+3)(p+4)} \qquad f(\infty)=0$$

$$F(p)=\frac{p+1}{p(p^2-1)} \qquad f(\infty)=1$$

$$F(p)=\frac{1}{p^3+2p^2+9p+68}$$

$$F(p)=\frac{p^3+p^2-2}{p^4+2p^3+2p^2-2p-3}$$

Table of Laplace transform pairs

$F(p)= L\{f(t)\}$	$f(t)$
1	$\delta(t)$
$\frac{1}{p}$	1
$\frac{1}{p^n}$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{1}{p-a}$	e^{at}
$\frac{1}{(p-a)^n}$	$\frac{t^{n-1}e^{at}}{(n-1)!}$
$\frac{\omega}{p^2+\omega^2}$	$\sin \omega t$
$\frac{p}{p^2+\omega^2}$	$\cos \omega t$

$F(p)= L\{f(t)\}$	$f(t)$
$\frac{a}{p^2-a^2}$	$\sinh at$
$\frac{p}{p^2-a^2}$	$\cosh at$
$\frac{2\omega p}{(p^2+\omega^2)^2}$	$t \sin \omega t$
$\frac{p^2-\omega^2}{(p^2+\omega^2)^2}$	$t \cos \omega t$
$\frac{\omega}{(p-a)^2+\omega^2}$	$e^{at} \sin \omega t$
$\frac{p-a}{(p-a)^2+\omega^2}$	$e^{at} \cos \omega t$

http://cam.zcu.cz/~danek/Students/ME4/slovniky/LT.pdf

Solved examples

1. $f(t) = 2 + 3t\mathrm{e}^{-2t} - 4t^2\mathrm{e}^{-3t}$

   $F(p) = \frac{2}{p} + \frac{3}{(p+2)^2} - \frac{4.2}{(p+3)^3}$

   $\mathrm{e}^{at}f(t) \triangleq F(p-a),\ a = -2,\ a = -3$

   $1 \triangleq \frac{1}{p},\quad t \triangleq \frac{1}{p^2},\ t^2 \triangleq \frac{2}{p^3}.$

2. $f(t) = 3\sin 2t - 5\cos 2t$

   $F(p) = \frac{3.2}{p^2+4} - \frac{5p}{p^2+4} = \frac{6-5p}{p^2+4}$

   $\sin 2t \triangleq \frac{2}{p^2+4},\ \cos 2t \triangleq \frac{p}{p^2+4}.$

3. $f(t) = 3t - \sin 2t$

   $F(p) = \frac{3}{p^2} - \frac{2}{p^2+4}$

   $t \triangleq \frac{1}{p^2},\ \sin 2t \triangleq \frac{2}{p^2+4}.$

4. $f(t) = t^2 - 1 + 3\mathrm{e}^{-t} + \cos 2t$

   $F(p) = \frac{2}{p^3} - \frac{1}{p} + \frac{3}{p+1} + \frac{p}{p^2+4}$

   $t^n \triangleq \frac{n!}{p^{n+1}},\ \mathrm{e}^{-2t} \triangleq \frac{1}{p+2},\ \cos 2t \triangleq \frac{p}{p^2+4}.$

5. $f(t) = 4\mathrm{e}^{t} + 2\mathrm{e}^{-3t} + \sin 2t$

   $F(p) = \frac{4}{p-1} + \frac{2}{p+3} + \frac{2}{p^2+4}$

   $\mathrm{e}^{at}f(t) \triangleq F(p-a),\ a = 1,\ a = -3 \quad 1 \triangleq \frac{1}{p},\ \sin 2t \triangleq \frac{2}{p^2+4}.$

6. $f(t) = (2t+5)\mathrm{e}^{-2t} + 3\cos t - 2\sin 3t$

   $F(p) = \frac{2}{(p+2)^2} + \frac{5}{p+2} + \frac{3p}{p^2+1} - \frac{6}{p^2+9}$

   $\mathrm{e}^{-2t}f(t) \triangleq F(p+2)$

   $1 \triangleq \frac{1}{p},\ t \triangleq \frac{1}{p^2},\ \cos t \triangleq \frac{p}{p^2+1},\ \sin 3t \triangleq \frac{3}{p^2+9}.$

7. $f(t) = t(\sin 2t + 4\cos 2t)$

   $F(p) = -\left(\frac{2}{p^2+4} + \frac{4p}{p^2+4}\right)' = \frac{4p^2+4p-16}{(p^2+4)^2}$

   $tf(t) \triangleq -F'(p) \qquad \sin 2t \triangleq \frac{2}{p^2+4},\ \cos 2t \triangleq \frac{p}{p^2+4}.$

8. $f(t) = (t+2)\cos 3t$

   $F(p) = -\left(\frac{p}{p^2+9}\right)' + \frac{2p}{p^2+9} = \frac{p^2-9}{(p^2+9)^2} + \frac{2p}{p^2+9} = \frac{2p^3+p^2+18p-9}{(p^2+9)^2}$

   $tf(t) \triangleq -F'(p) \quad \cos 3t \triangleq \frac{p}{p^2+9}.$

9. $f(t) = (3t^2 + 2t - 1)\mathrm{e}^{-t} + (t+1)\sin 2t$

   $F(p) = \frac{6}{(p+1)^3} + \frac{2}{(p+1)^2} - \frac{1}{p+1} - \left(\frac{2}{p^2+4}\right)' + \frac{2}{p^2+4} = \frac{6}{(p+1)^3} + \frac{2}{(p+1)^2} - \frac{1}{p+1} + \frac{4p}{(p^2+4)^2} + \frac{2}{p^2+4}$

   $\mathrm{e}^{-t}f(t) \triangleq F(p+1),\ tf(t) \triangleq -F'(p)$

   $1 \triangleq \frac{1}{p},\ t \triangleq \frac{1}{p^2},\ t^2 \triangleq \frac{2}{p^3},\ \sin 2t \triangleq \frac{2}{p^2+4}.$

10. $f(t) = t\mathrm{e}^{-3t} + (t-5)\cos 3t$

    $F(p) = \frac{1}{(p+3)^2} - \left(\frac{p}{p^2+9}\right)' - \frac{5p}{p^2+9} = \frac{1}{(p+3)^2} + \frac{p^2-9}{(p^2+9)^2} - \frac{5p}{p^2+9}$

    $\mathrm{e}^{-3t}f(t) \triangleq F(p+3),\ tf(t) \triangleq -F'(p)$ a vzorce $t \triangleq \frac{1}{p^2},\ \cos 3t \triangleq \frac{p}{p^2+9}.$

https://math.feld.cvut.cz/prucha/m2c/ltru.pdf

Revision

1.Write the definition formula of L-transform.

2. Do you remember Laplace transform of this following functions ?

$$L\left\{1(t)\right\} =$$

$$L\left\{e^{-at}\right\} =$$

$$L\left\{\delta(t)\right\} =$$

$$L\left\{f'(t)\right\} =$$

$$L\left\{\int_0^\infty f(t)\,dt\right\} =$$

Revision

3. Write L-transform of

- the unit step
- a constant function
- the ramp function
- an exponential function
- Dirac function
- derivation: $L\{ x'(t)\} =$
- function shifted in time: $L\{f(t-a)\} =$

References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.
ISBN: 81-7409-107-6

[2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing

[3] Matlab&Simulink: R2015a

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2. Inverse Laplace Transform

Inverse Laplace transform of rational function

Find the inverse Laplace transform of the following function using partial fraction expansion:

1. <u>Function has distinct real roots:</u>

$$F(p) = \frac{6p^2 - 12}{(p+1)(p+2)(p-2)} = \frac{A}{p+1} + \frac{B}{p+2} + \frac{C}{p-2}$$

$$A = \lim_{p \to -1} (p+1) \cdot F(p) = \lim_{p \to -1} \frac{6p^2 - 12}{(p+2)(p-2)} = \frac{-6}{-3} = 2$$

$$B = \lim_{p \to -2} (p+2) \cdot F(p) = \lim_{p \to -2} \frac{6p^2 - 12}{(p+1)(p-2)} = 3$$

$$C = \lim_{p \to 2} (p-2) \cdot F(p) = \lim_{p \to 2} \frac{6p^2 - 12}{(p+2)(p+1)} = 1$$

$$F(p) = \frac{6p^2 - 12}{(p+1)(p+2)(p-2)} = \frac{2}{p+1} + \frac{3}{p+2} + \frac{1}{p-2}$$

Then the inverse L-transform of function is the sum of inverse L-transform of particular fractions.

$$f(t) = 2 \cdot e^{-t} + 3 \cdot e^{-2t} + e^{+2t}$$

(this is the sum of three exponential functions)

Inverse Laplace transform of rational function

Find the inverse Laplace transform of the **following** function using partial fraction expansion:

2. Function has two complex conjuated roots:

$$F(p) = \frac{p+5}{p^2+4p+13}$$

Roots can be found using Matlab command: >> p = *roots(num, den)*

$p_{1,2}$ = -2 ± 3j

$$F(p) = \frac{A}{(p+2-3j)} + \frac{B}{(p+2+3j)}$$

$$A = \lim_{p \to -2+3j} (p+2-3j)\frac{p+5}{(p+2-3j)(p+2+3j)} = \frac{-2+3j+5}{-2+3j+2+3j} = \frac{3+3j}{6j} = \frac{3-3j}{6} = \frac{1}{2}(1-j)$$

$$B = \frac{1}{2}(1+j)$$

Computation of coefficient B is not necessary, because it must equal to the complex-conjugate of A.

Then:

$$F(p)=\frac{\frac{1}{2}(1-j)}{(p+2-3j)}+\frac{\frac{1}{2}(1+j)}{(p+2+3j)}$$

Then for $t > 0$:

$$f(t)=\frac{1}{2}(1-j)\cdot e^{(-2+3j)\cdot t}+\frac{1}{2}(1+j)\cdot e^{(-2-3j)\cdot t}=\frac{1}{2}\cdot e^{-2t}\cdot\left[2\cdot(\cos 3t+\sin 3t)\right]$$

You can see that the function f(t) is the sum of two harmonic functions.

$$f(t)=\frac{1}{2}(1-j)\cdot e^{(-2+3j)\cdot t}+\frac{1}{2}(1+j)\cdot e^{(-2-3j)\cdot t}=\frac{1}{2}\cdot e^{-2t}\left[(1-j)\cdot e^{3jt}+(1+j)\cdot e^{-3jt}\right]=$$

$$=\frac{1}{2}e^{-2t}\left(-je^{3jt}+je^{-3jt}+e^{3jt}+e^{-3jt}\right)=\frac{1}{2}e^{2t}\left[2\frac{1}{2j}\left(e^{j3t}-e^{-j3t}\right)+2\frac{1}{2}\left(e^{j3t}+e^{-j3t}\right)\right]=$$

$$\boxed{=\frac{1}{2}e^{-2t}\left(2\sin 3t+2\cos 3t\right)}$$

Inverse Laplace transform of rational function.

Find the inverse Laplace transform of the following function using partial fraction expansion:

3. <u>Function has repeated roots:</u>

$$F(p) = \frac{2}{(p+1)^3(p+2)} = \frac{A_1}{(p+1)} + \frac{A_2}{(p+1)^2} + \frac{A_3}{(p+1)^3} + \frac{B}{p+2}$$

We can state the general formula for computing A_i.

$$A_i = \frac{1}{(m-i)!} \lim_{p \to p_i} \frac{d^{m-1}}{dp^{m-1}} \left[(p-p_i)^m \cdot F(p) \right]$$

where m = 3 (which is the number of repeated roots)

$$A_1 = \frac{1}{(3-1)!}\lim_{p\to -1}\frac{d^2}{dp^2}\left[(p+1)^3 F(p)\right] = \frac{1}{2}\lim_{p\to -1}\frac{d^2}{dp^2}\frac{2}{(p+2)} = \frac{1}{2}\lim_{p\to -1}\frac{+4}{(p+2)^3} = +2$$

$$A_2 = \frac{1}{(3-2)!}\lim_{p\to -1}\frac{d}{dp}\left[(p+1)^3 F(p)\right] = \lim_{p\to -1}\frac{d}{dp}\frac{2}{(p+2)} = \lim_{p\to -1}\frac{-2}{(p+2)^2} = -2$$

$$A_3 = \lim_{p\to -1}(p+1)^3 F(p) = 2$$

$$B = \lim_{p\to -2}(p+2)F(p) = -2$$

Then: $$F(p) = \frac{2}{(p+1)} + \frac{-2}{(p+1)^2} + \frac{2}{(p+1)^3} + \frac{-2}{p+2} \quad \Rightarrow$$

$$f(t) = 2\cdot e^{-t} - 2t\cdot e^{-t} + 2\frac{t^2}{2}\cdot e^{-t} - 2\cdot e^{-2t} = 2\left[\left(1 - t + \frac{t^2}{2}\right)\cdot e^{-t} - e^{-2t}\right]$$

Problem solving:

Find the inverse Laplace transform of the following functions:

a) $F(p) = \dfrac{6p}{(p+1)(p+2)(p-2)}$

b) $F(p) = \dfrac{6}{(p+1)(p+2)(p+3)}$

Answer :

a) $2.e^{-t} - 3.e^{-2t} + e^{+2t}$

b) $3.e^{-t} - 6.e^{-2t} + 3.e^{-3t}$

Problem solving:

Find the inverse Laplace transform of the following functions:

a) $F(p) = \dfrac{1}{p(p+1)}$

b) $F(p) = \dfrac{1}{p^2 + 4p + 8}$

c) $F(p) = \dfrac{5}{p \cdot (p^2 + 4p + 5)}$

d) $F(p) = \dfrac{p+6}{p \cdot (p^2 + 4p + 3)}$

Solution:

a) $f(t) = 1 - e^{-t}$

b) $f(t) = \frac{1}{2} e^{-2t} sin(2t)$

c) $f(t) = 1 - e^{-2t}\cos t - 2e^{-2t}\sin t$

d) $f(t) = 2 - 2{,}5e^{-t} + \frac{1}{2} e^{-3t}$

Differential equation solving using Laplace Transform

Find the L-transform of the differential equation given below and hence evaluate the time solution of the it, when the initial conditions are y(0) = 0, y'(0) = 6.

$$y''(t) + 5y'(t) + 6y(t) = 12e^{t}$$

Solution in L-transform:

$$p^2 \cdot Y(p) - p \cdot y(0) - y(0) \quad + \quad 5 \cdot \left(p \cdot Y(p) - y(0)\right) \quad + \quad 6 \cdot Y(p) = \frac{12}{p-1}$$

$$p^2 \cdot Y(p) - 6 + 5 \cdot p \cdot Y(p) + 6 \cdot Y(p) = \frac{12}{p-1}$$

$$Y(p) \cdot \left(p^2 + 5p + 6\right) = \frac{6p+6}{p-1}$$

$$Y(p) = \frac{6p+6}{(p-1)\left(p^2+5p+6\right)}$$

Solution in time domain:

$$y(t) = L^{-1}\{Y(p)\} = L^{-1}\left\{\frac{6 \cdot p + 6}{(p-1)\left(p^2 + 5p + 6\right)}\right\} = L^{-1}\left\{\frac{1}{p-1} + \frac{2}{p+2} - \frac{3}{p+3}\right\}$$

$$y(t) = e^{+t} + 2e^{-2t} - 3e^{-3t}$$

Obtain the solution of diferential equation given below:

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 0$$ given *x(0)=0* and *x'(0)=1*

$$L\left\{\frac{d^2x}{dt^2}\right\} + L\left\{2\frac{dx}{dt}\right\} + L[2x] = 0$$

$$[p^2X(p) - p \cdot x(0) - x(0)] + [2pX(p) - x(0)] + 2X(p) = 0$$

$$X(p) = \frac{1}{p^2 + 2p + 2} = \frac{1}{(p+1)^2 + 1^2}$$

$$L^{-1}\{X(p)\} = L^{-1}\left\{\frac{1}{p^2 + 2p + 2}\right\} = L^{-1}\left\{\frac{1}{(p+1)^2 + 1^2}\right\}$$

$$x(t) = e^{-1} \cdot sin\ t$$

Obtain the solution of diferential equation given below:

$$2\frac{dx}{dt} + 8x = 10; \quad given\, x(0) = 2$$

Taking L-transform of both sides the following equation is obtained:

$$L\left\{2\frac{dx}{dt} + 8x\right\} = L\{10\}$$

$$2[pX(p) - x(0)] + 8X(p) = \frac{10}{p}$$

Substituting $x(0) = 2$

$$2[pX(p) - 2] + 8X(p) = \frac{10}{p}$$

Simplifying:

$$X(p) = \frac{2p + 5}{p(p + 4)}$$

Using partial fraction expansion:

$$X(p) = \frac{k_1}{p} + \frac{k_2}{p + 4} = \frac{1{,}25}{p} + \frac{0{,}75}{p + 4}$$

Taking inverse L-transform of both sides:

$$x(t) = 1{,}25 + 0{,}75e^{-4t}$$

Problem solving

Find the inverse Laplace transform of the following functions using Matlab.

- $f1(t) = e^{-8t}$
- $f2(t) = sin(-8t)$
- $f3(t) = cos(-8t)$
- $f4(t) = 10$
- $f5(t) = t$
- $f6(t) = t2$

References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.
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[2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing

[3] Matlab&Simulink: R2015a

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3. Introduction to Matlab

Matrix operations

We can do matrix arithmetic using the standard arithmetical operators.

Addition

```
>> A=[1 2;3 4]
A =
    1    2
    3    4
>> B=[2 2; 3 3]
B =
    2    2
    3    3
```

\>> **C=A+B**

```
C =
   3    4
   6    7
```

Substraction

\>> **C1=A-B**

```
C1 =
        1    0
        0    1
```

Matrix operations

We can do matrix arithmetic using the standard arithmetical operators.

Multiplicative operations - based on definition

```
>> A=[1 2;3 4]
A =
    1    2
    3    4
>> B=[2 2; 3 3]
B =
    2    2
    3    3
>> C=A*B
C =
    8    8
   18   18
```

Multiplicative operations- element-by-element

```
>> C1=A.*B
C1 =
    2    4
    9   12
```

Matrix operations

We can do matrix arithmetic using the standard arithmetical operators.

Division operations - based on definition

>> A=[1 2;3 4];

>> B=[2 2; 3 1];

- A\B backslash or left matrix divide

It is the same as $A^{-1} * B$

>> **C=A\B**

C =

-1.0000 -3.0000

1.5000 2.5000

- A/B Slash or right matrix divide.

It is the same as $\mathrm{A} * B^{-1}$

>> **C1=A/B**

C1 =

1.2500 -0.5000

2.2500 -0.5000

Division operations- element-by-element

>> **C2=A.\B**

C2 =

2.0000 1.0000

1.0000 0.2500

>> C3=A./B

C3 =

0.5000 1.0000

1.0000 4.0000

Matrix functions - numerical linear algebra.

1. **Rank - Rank of matrix**

returns the number of singular values of A

>> k = rank(A)

2. **Det - Matrix determinant**

returns the determinant of square matrix A.

>> d = det(A)

3. **Trace - Sum of diagonal elements**

is the sum of the diagonal elements of the matrix A.

>> b = trace(A)

4. **Inv - Matrix inverse**

returns the inverse of the square matrix X.

>> Y = inv(X)

Matrix functions - numerical linear algebra.

5. Eig - Eigenvalues and eigenvectors

returns a column vector containing the eigenvalues of square matrix A.

```
>>  e = eig(A)
```

6. Poly - Polynomial with specified roots

This MATLAB function where A is an n-by-n matrix returns an n+1 element row vector whose elements are the coefficients of the characteristic polynomial -> $\det(\lambda I - A)$.

```
>> p = poly(A)
```

7. Matrix transposed

```
>>  A_transposed = A'
```

Polynomials

Representing polynomials:

Matlab software represents polynomials as row vectors containing coefficients ordered by descending powers. For example, consider the equation:

$p(x) = x^3 - 2x - 5$

To enter this polynomial into MATLAB, use:

```
>> p=[1 0 -2 -5]
p =
    1     0    -2    -5
```

A coefficient of 0 indicates an intermediate power that is not present in the equation.

Evaluating polynomials

Polyval – the function evaluates polynomial at specified value. To evaluate p at p=5, use

```
>> polyval(p,5)
ans =
   110
```

Polynomial roots

Roots – the function solves polynomial equations of the form $a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0 = 0$

Polynomial equations contain a single variable with nonnegative exponents.

This function returns the roots of the polynomial represented by p as a column vector.

Example: $p = x^3 - 2x - 5$

```
>> p=[1 0 -2 -5];
>> roots(p)

ans =

   2.0946 + 0.0000i
  -1.0473 + 1.1359i
  -1.0473 - 1.1359i
```

Residue - Convert between partial fraction expansion and ratio of two polynomials

[r,p,k] = residue(b,a) finds the residues, poles, and direct term of a Partial Fraction Expansion of the ratio of two polynomials, where the expansion is of the form

$$\frac{b(b)}{a(p)} = \frac{b_m p^m + b_{m-1} p^{m-1} + \cdots + b_1 p + b_0}{a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0}$$

The inputs to residue are vectors of coefficients of the polynomials:

$b = [b_m \; b_m \; \ldots b_1 \; b_0]$ and $a = [a_n \; a_{n-1} \; \ldots a_1 \; a_0]$

The outputs are the residues $r = [r_n \; r_{n-1} \; \ldots r_1]$, the poles $p = [p_n \; p_{n-1} \; \ldots p_1]$, and the polynomial k.

```
>> b=1;
>> a=[1 5 6];
>> [r,p,k]=residue(b,a)
r =
  -1.0000
   1.0000
p =
  -3.0000
  -2.0000
k =
    []
```

[b,a] = residue(r,p,k) converts the partial fraction expansion back to the ratio of two polynomials and returns the coefficients in b and a.

Conv - **Convolution and polynomial multiplication**

returns the convolution of vectors u and v.

Example: $(x+2)(2x^2+4x+1) = 2x^3+8x^2+9x+2$

```
>> u=[1 2];

>> v=[2 4 1]

v =

    2    4    1

>> conv(u,v)

ans =

  2    8    9    2
```

Problems

Find the inverse Laplace transform of the **following** function using Matlab:

1. $$F(p)=\frac{p+5}{p^2+4p+13}$$

2. $$F(p)=\frac{6p^2-12}{(p+1)(p+2)(p-2)}=\frac{A}{p+1}+\frac{B}{p+2}+\frac{C}{p-2}$$

3. $$F(p)=\frac{2}{(p+1)^3(p+2)}=\frac{A_1}{(p+1)}+\frac{A_2}{(p+1)^2}+\frac{A_3}{(p+1)^3}+\frac{B}{p+2}$$

Problem solving:

Find the inverse Laplace transform of the following functions using Matlab:

a) $F(p) = \dfrac{6p}{(p+1)(p+2)(p-2)}$

b) $F(p) = \dfrac{6}{(p+1)(p+2)(p+3)}$

Answer :

a) $2.e^{-t} - 3.e^{-2t} + e^{+2t}$

b) $3.e^{-t} - 6.e^{-2t} + 3.e^{-3t}$

References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.
ISBN: 81-7409-107-6

[2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing

[3] Matlab&Simulink: R2015a

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External Description of Dynamic Systems

Problems

For the transfer function plot the poles and zeros in *p*-plane.

$$G(p) = \frac{1}{2}\frac{\left(p^2+4\right)\left(1+2{,}5\,p\right)}{\left(p^2+2\right)\left(1+0{,}5\,p\right)}$$

2. The dynamic system is done by differential equation. Find the transfer function *G(p)*.

$$5y''-2y'+3y = u''+4u'+4u$$

3. Find zeros and poles of dynamic system which are done by transfer function. Plot it in *p*-plane.

$$G(p) = \frac{p^2+4p+4}{5p^3-2p+3}$$

Problems

System has the transfer function: $G(p) = \dfrac{1}{p^2 + 5p + 6}$

1. Compute the impulse response *g(t)* in time domain.
2. Compute the step response *H(p)* and in time domain *h(t)*.
3. Compute steady-state value (final value) for both responses using the limits theorems.
4. Compute step response *h(t)* from *g(t)* and impulse response *g(t)* from *h(t)*.
5. Simulate the response using Matlab and Matlab-Simulink.

1. Impulse response:

$$g(t) = L^{-1}\{\, G(p)\,\} = L^{-1}\left\{ \frac{1}{p^2 + 5p + 6} \right\} = L^{-1}\left\{ \frac{1}{(p+3)(p+2)} \right\} = L^{-1}\left\{ \frac{-1}{(p+3)} + \frac{1}{(p+2)} \right\} = -e^{-3t} + e^{-2t}$$

2. Step response:

$$h(t) = L^{-1}\left\{ G(p)\cdot\frac{1}{p} \right\} = L^{-1}\{\, H(p)\,\} = L^{-1}\left\{ \frac{1}{\left(p^2 + 5p + 6\right)p} \right\} = L^{-1}\left\{ \frac{\frac{1}{3}}{(p+3)} + \frac{-\frac{1}{2}}{(p+2)} + \frac{\frac{1}{6}}{p} \right\} = \frac{1}{3}e^{-3t} - \frac{1}{2}e^{-2t} + \frac{1}{6}$$

3. Final value of the impulse response

$$g(t \to \infty) = \lim_{p \to 0} p \cdot G(p) = \lim_{p \to 0} p \frac{1}{p^2 + 5p + 6} = 0$$

Verification in time domain: $\lim\limits_{p \to \infty} g(t) = 0$

4. Final value of the step response

$$h(t \to \infty) = \lim_{p \to 0} p \cdot H(p) = \lim_{p \to 0} p \frac{1}{\left(p^2 + 5p + 6\right)p} = \frac{1}{6}$$

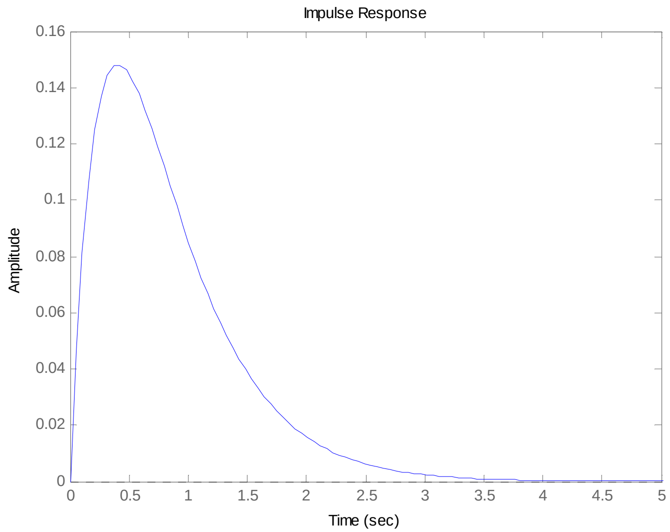
Verification in time domain: $\lim\limits_{p \to \infty} h(t) = \frac{1}{6}$

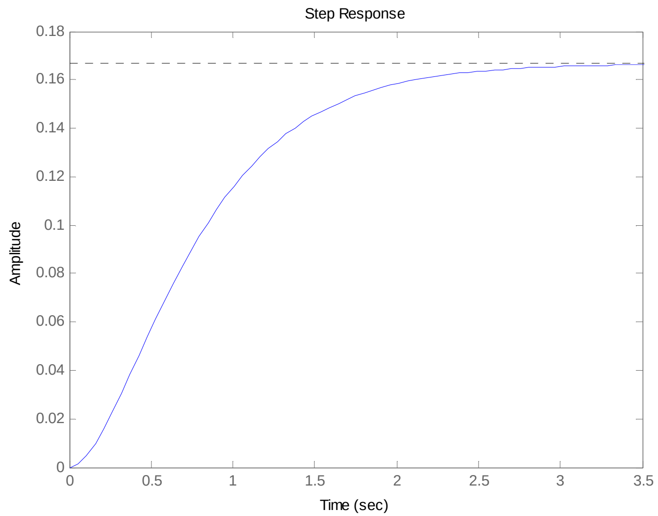
5. Relation between impulse response and step response:

$$h(t) = \int_0^t g(t)\,dt = \int_0^t \left(-e^{-3t} + e^{-2t}\right) dt = \left[\frac{1}{3} - e^{-3t} - \frac{1}{2}e^{-2t}\right]_0^t = \frac{1}{3} - e^{-3t} - \frac{1}{2}e^{-2t} - \frac{1}{3} + \frac{1}{2}$$

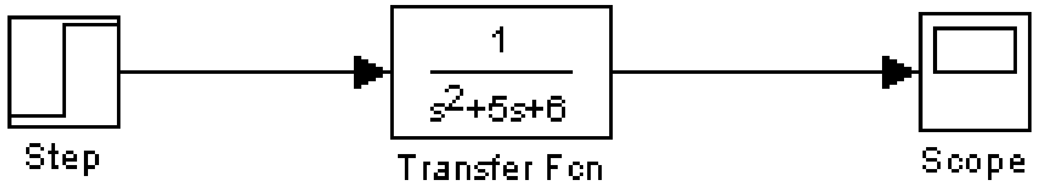
$$g(t) = \frac{d\,h(t)}{d\,t} = \frac{d}{dt}\left(\frac{1}{3} - e^{-3t} - \frac{1}{2}e^{-2t} + \frac{1}{6}\right) = e^{-3t} - e^{-2t}$$

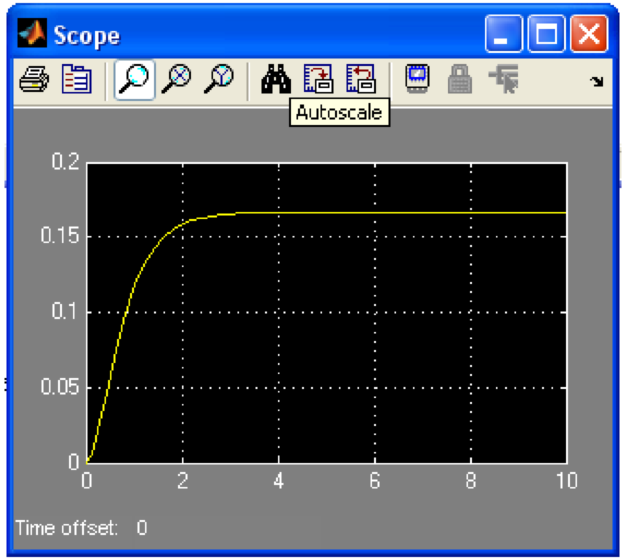
6. Simulation using Matlab.





Simulation of the step response using Simulink:





Problem: Nyquist plot

Procedure for mapping *G(p)* from *p*-plane to *G(jω)*-plane.

For plotting *G(jω)* the independent variable *p* is varied on the entire imaginary axis from *ω*=+0 to *ω*=+∞.

The plot variation from *ω* = -0 to *ω* = - ∞ use vertical image of the plot from *ω* = +0 to *ω* = +∞.

The examples given below illustrates the process of Nyquist plot.

$$G(p) = \frac{100}{p^2 + p + 20}$$

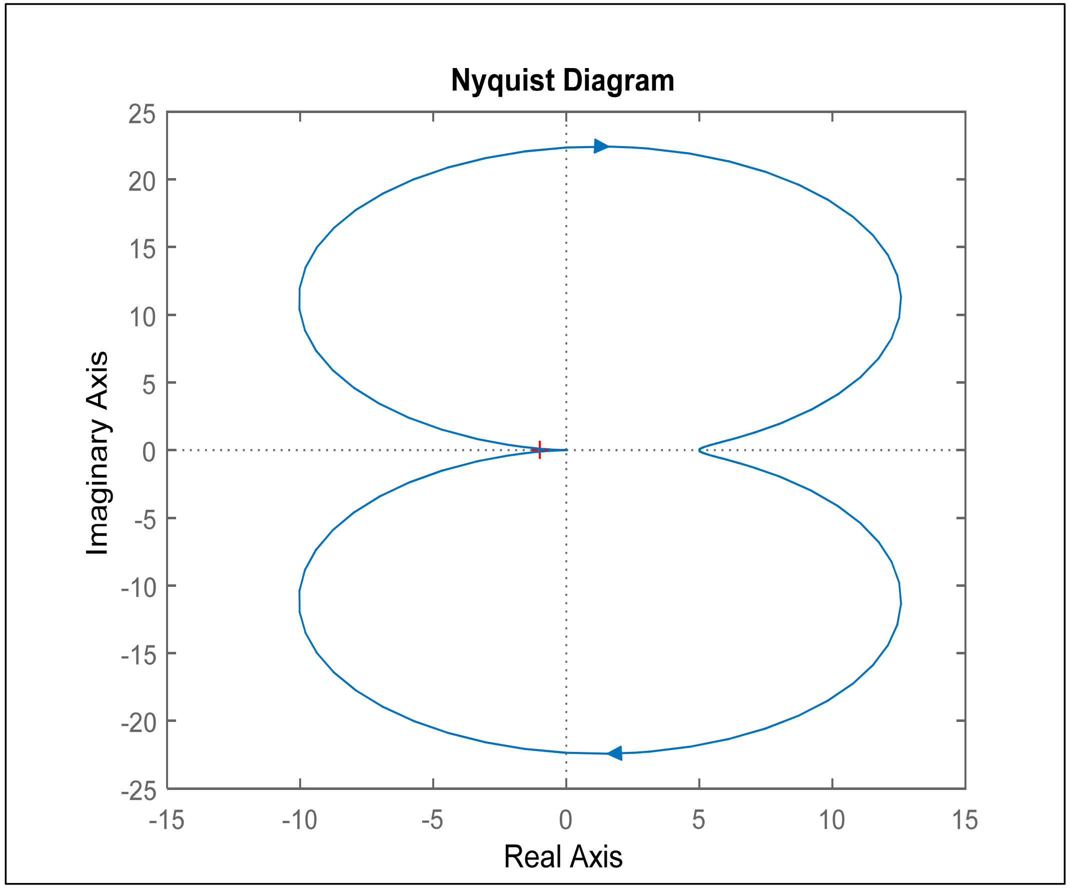
$$G(j\omega) = \frac{100}{(j\omega)^2 + j\omega + 20}$$

$$G(j\omega) = \frac{100}{(j\omega)^2 + j\omega + 20} = \frac{100}{-\omega^2 + j\omega + 20} = \frac{100}{(20-\omega^2) + j\omega} = \frac{100}{(20-\omega^2) + j\omega} \cdot \frac{(20-\omega^2) + j\omega}{(20-\omega^2) + j\omega}$$
$$= \frac{100(20-\omega^2) - 100j\omega}{400 - 39\omega^2 + \omega^4}$$

$$Re\{G(j\omega)\} = \frac{(20-\omega^2)}{400 - 39\omega^2 + \omega^4} \cdot 100 \qquad Im\{G(j\omega)\} = \frac{-\omega}{400 - 39\omega^2 + \omega^4} \cdot 100$$

Problem: Nyquist plot

ω	$Re\{G(j\omega)\}$	$Im\{G(j\omega)\}$
0	5	0
1	5.25	-0.28
2	6.15	-0.77
3	8.46	-2.31
4	12.5	-12.5
$\sqrt{20}$	0	-22
5	-10	-10
6	-5.48	-2.05
8	-2.2	-0.4
10	-1.23	-0.15
∞	0	0



Problem: Bode plot

Sketch the Bode plot for transfer function: $G(p) = \dfrac{50}{(p+1)(p+2)}$

Solution:

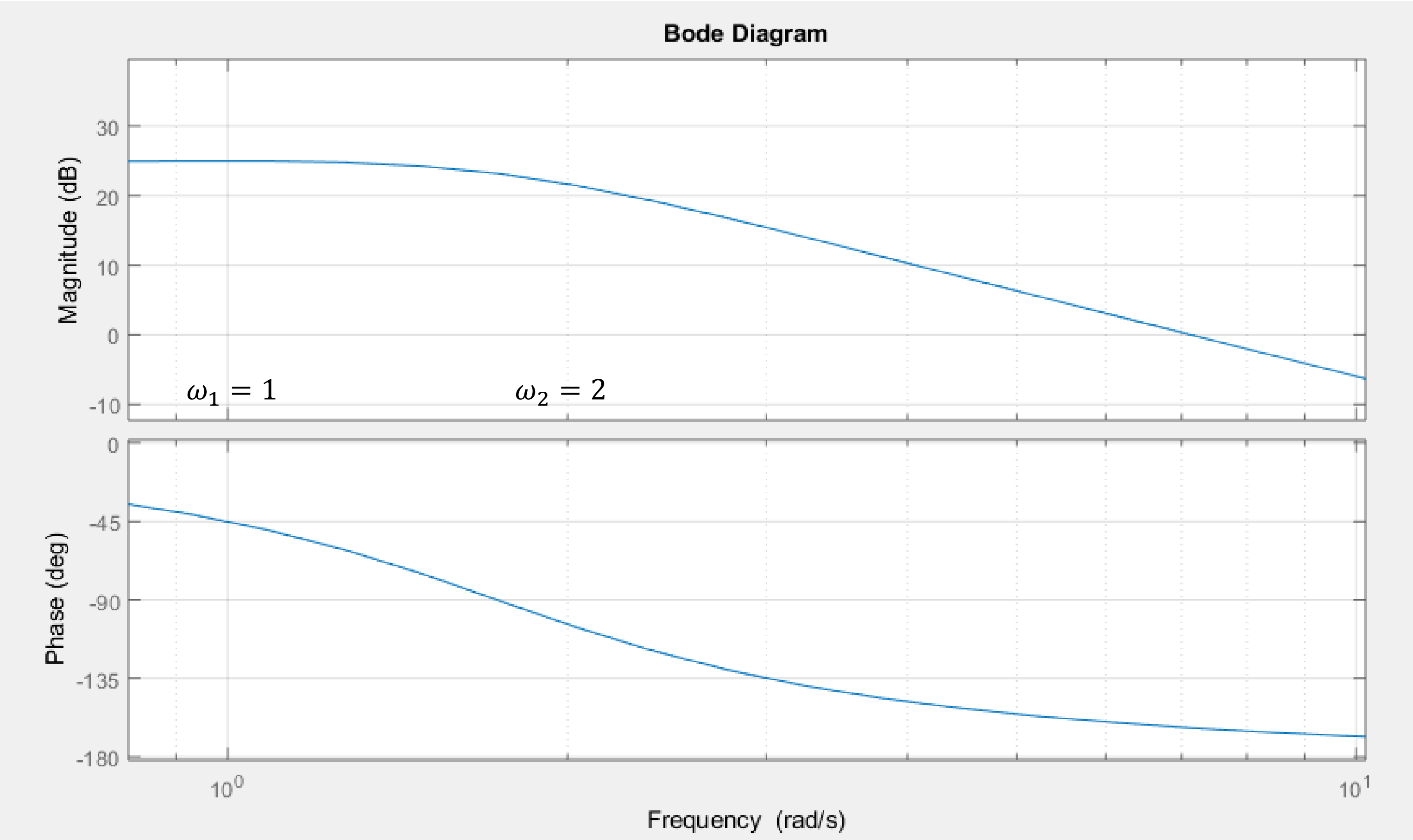
- put *p=jω* and rewrite the transfer function in Bode form: $G(j\omega) = \dfrac{50/2}{(j\omega+1)(0.5j\omega+1)}$
- Time constants: $T_1 = 1 \quad T_2 = 0.5$
- Magnitude in decibel: $|G(j\omega)|_{db} = 20\log_{10} 25 - 20\log_{10}|j\omega+1| - 20\log_{10}|0.5j\omega+1|$
- Initial magnitude: $A_{db} = 20\log_{10} 25 = 27{,}9$
- Corner frequency of the asymptotic plot: $\omega_1 = \dfrac{1}{T_1} = 1$

$$\omega_2 = \frac{1}{T_2} = 2$$

- Phase angel: $\varphi(\omega) = -\arctan(\omega) - \arctan(0.5\omega)$

Bode plot

$$G(p) = \frac{50}{(p+1)(p+2)}$$



References

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Types of Controlled Systems

Problem: Higher-order system's approximation

Find the simplest lower-order approximation of the following transfer function:

$$G(p) = \frac{3}{(0{,}1p+1)(0{,}5p+1)(p+1)(3p+1)}$$

The largest time constant: T=3 ,

The corresponding dominant pole: p = -1/3

Transport delay : T_d = $\sum T_i$ = 0,1 + 0,5 + 1 =1,

We may approximate as: $$G(p) = \frac{3\,e^{-1{,}6\,p}}{(3p+1)}$$

Transport delay approximation

A time delay is also called transport delay or dead time or transport lag. To handle the time delay we use so-called **Pade' approximation** which expresses the function as a ratio of two polynomials.

The simplest is the first-order Pade' approximation:

$$e^{-T_d\,p} = \frac{1-\frac{T_d}{2}p}{1+\frac{T_d}{2}p}$$

The second-order Pade' approximation:

$$e^{-T_d \cdot p} = \frac{T_d^2 \cdot p^2 - 6T_d\,p + 12}{T_d^2 \cdot p^2 + 6T_d\,p + 12}$$

Problem: Transport delay approximation

Use the first order Pade' app. To plot the unit-step response for the first order system with dead-time function.

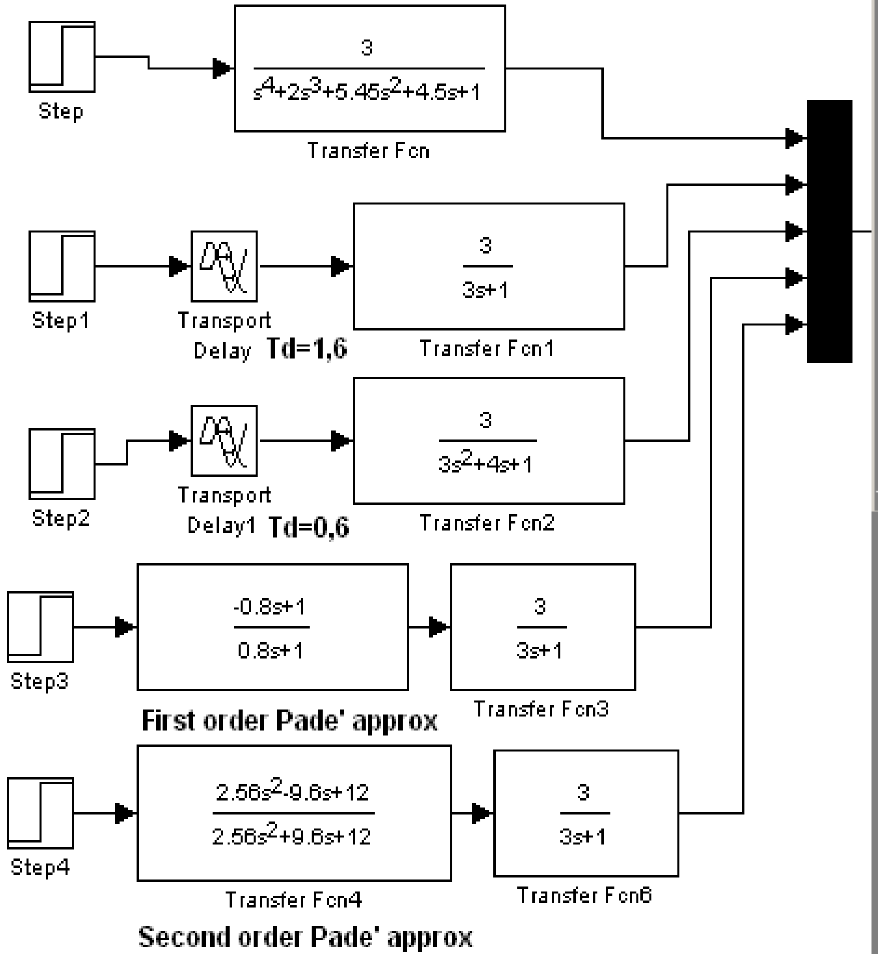
$$G(p)=\frac{e^{-3p}}{10p+1}$$

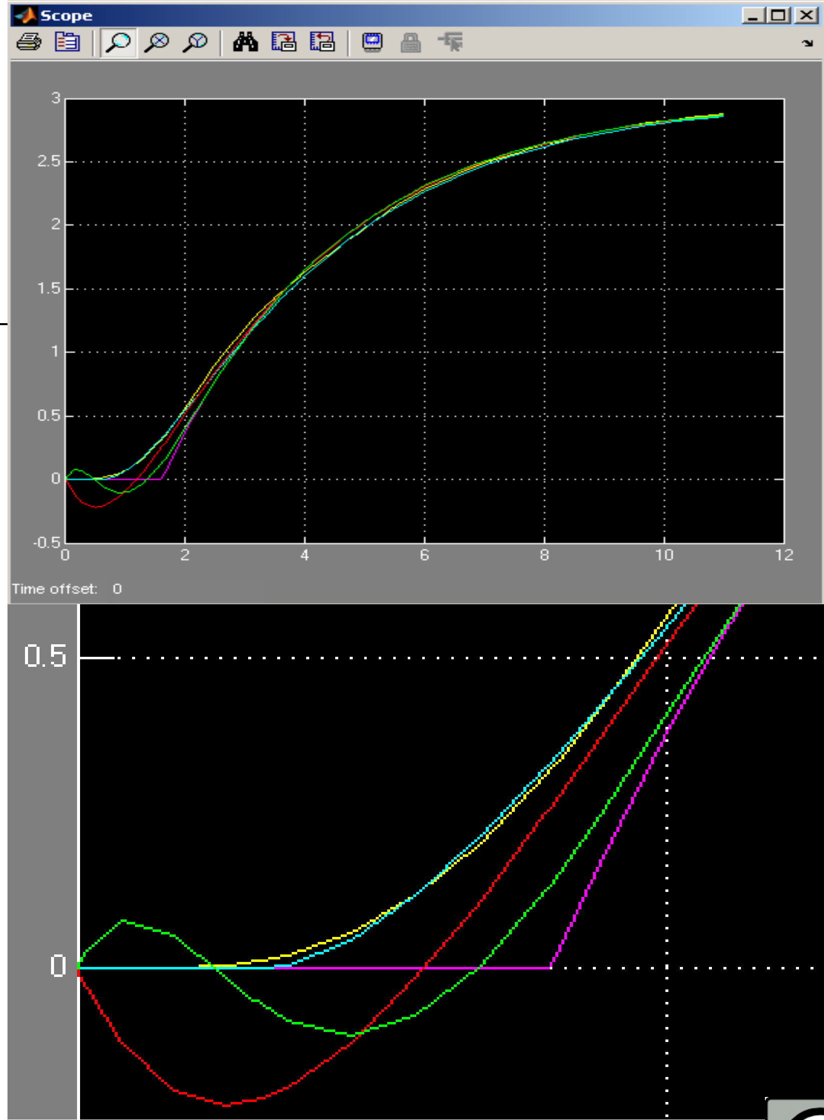
Solution:

$$e^{-T_d\,p} = \frac{1-\frac{T_d}{2}p}{1+\frac{T_d}{2}p}=\frac{1-\frac{3}{2}p}{1+\frac{3}{2}p}=\frac{-1{,}5p+1}{1{,}5p+1}$$

$$G(p)=\frac{1}{10p+1}\,\frac{-1{,}5p+1}{1{,}5p+1}$$

Comparison: Time delay approximation and Pade' approximation





Individual project

EXTERNAL DESCRIPTION OF SYSTEM

The system is given by differential equation:

$$y'' + 6\,y' + 8\,y = 3\,u' + 2\,u \qquad (0)$$

- Compute the transfer function of this system.
- Compute the impulse response g(t) in time domain.
- Compute the step response H(p) and h(t) in time domain.
- Compute steady-state value (final value) for both responses using the limits theorems.
- Draw the pole-zero configuration in complex plane.
- Simulate the responses using Matlab.

Individual project

EXTERNAL DESCRIPTION OF SYSTEM

The system is given by differential equation:

(1) $y'' + 7y' + 12y = 4u' + u$

(2) $y'' + 7y' + 10y = 2u' + 3u$

(3) $y'' + 8y' + 15y = 3u'$

(4) $y'' + 9y' + 20y = 3u' + 2u$

(5) $y'' + 7y' + 6y = 4u' + u$

(6) $y'' + 8y' + 12y = 2u' + 3u$

(7) $y'' + 9y' + 18y = 3u'$

(8) $y'' + 10y' + 24y = 3u' + 2u$

(9) $y'' + 8y' + 7y = 4u' + u$

(10) $y'' + 9y' + 14y = 2u' + 3u$

(11) $y'' + 10y' + 21y = 3u'$

(12) $y'' + 10y' + 24y = u' + u$

(13) $y'' + 11y' + 30y = 2u' + u$

Individual project

EXTERNAL DESCRIPTION OF SYSTEM

The system is given by differential equation:

(14) $y'' + 11\,y' + 28\,y = 2\,u'$
(15) $y'' + 5\,y' + 4\,y = 2\,u'' + u$
(16) $y'' + 6\,y' + 5\,y = 4\,u'' + u$
(17) $y'' + 10\,y' + 24\,y = 3u'' + u$
(18) $y'' + 6\,y' + 8\,y = 5u'' + u$
(19) $y'' + 7\,y' + 12\,y = u'' + u$
(20) $y'' + 5\,y' + 6\,y = 2u''$
(21) $y'' + 7\,y' + 10\,y = 3\,u'' + u$

References

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State-space Representation of Controlled Dynamic Systems (Internal Description)

Frobenius' canonical form of a state space description

$$G(p) = \frac{Y(p)}{U(p)} = \frac{2p + 1}{p^3 + 9p^2 + 26p + 24}$$

The corresponding differential equation for the transfer function above is of third order; therefore we need three state variables. The state equations are chosen below:

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = x_3(t)$$

$$\dot{x}_3(t) = -24x_1(t) - 26x_2(t) - 9x_3(t) + u(t)$$

And the output is: $y(t) = x_1(t) + 2x_2(t)$

Frobenius' canonical form of state space description

The state model **in matrix form** is:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix} \cdot \begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} \cdot u(t)$$

$$y(t) = \begin{bmatrix}1 & 2 & 0\end{bmatrix} \cdot \begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

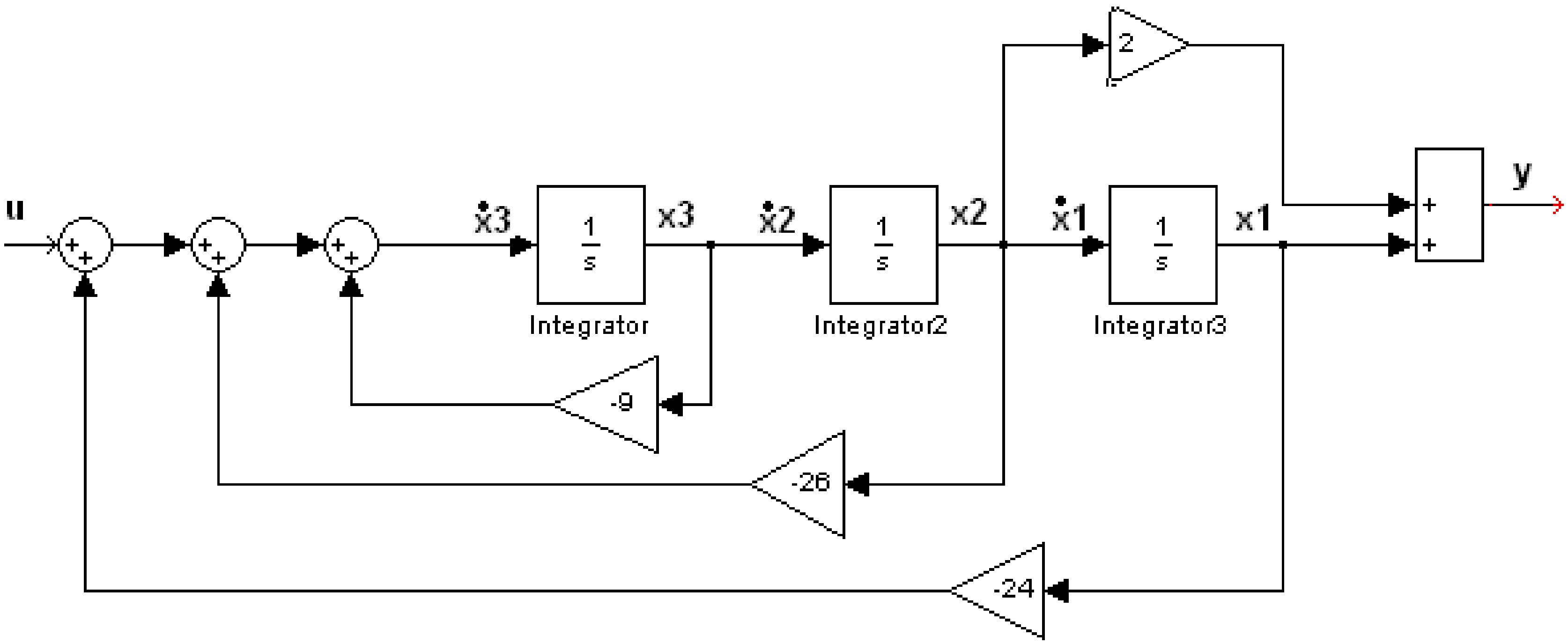
$$A = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix} \qquad B = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} \qquad C = \begin{bmatrix}1 & 2 & 0\end{bmatrix} \qquad D = [\,0\,]$$

Frobenius' canonical form of state space description

To obtain the block diagram we need three integrators and several summing junctions. The output of each integrator is assigned to a state variable.

Block diagram of Frobenius' form in relation to transfer function:

$$G(p)=\frac{2p+1}{p^3+9p^2+26p+24}$$



Jordan's canonical form of state space description

Jordan's decomposition is carried out by splitting the given transfer function into partial fraction.

$$G(p) = \frac{Y(p)}{U(p)} = \frac{2p+1}{p^3+9p^2+26p+24} = \frac{1/2}{p+2} + \frac{-1}{p+3} + \frac{1/2}{p+4}$$

We define the state variables x_1, x_2, x_3 as follows

$$\frac{X_1(p)}{U(p)} = \frac{1}{p+2} \Rightarrow \dot{x}_1(t) + 2x_1(t) = u(t)$$

$$\frac{X_2(p)}{U(p)} = \frac{1}{p+3} \Rightarrow \dot{x}_2(t) + 3x_2(t) = u(t)$$

$$\frac{X_3(p)}{U(p)} = \frac{1}{p+4} \Rightarrow \dot{x}_3(t) + 4x_3(t) = u(t)$$

$\rightarrow$

$$\boxed{\begin{aligned} \dot{x}_1(t) &= -2x_1(t) + u(t) \\ \dot{x}_2(t) &= -3x_2(t) + u(t) \\ \dot{x}_3(t) &= -4x_3(t) + u(t) \end{aligned}}$$

Jordan's canonical form of state space description

and output variable *y(t)* as follows:

$$G(p)=\frac{Y(p)}{U(p)}=\frac{1}{2}\frac{X_1(p)}{U(p)} - \frac{X_2(p)}{U(p)}+\frac{1}{2}\frac{X_3(p)}{U(p)} \Rightarrow Y(p)=\frac{1}{2}X_1(p)-X_2(p)+\frac{1}{2}X_3(p)$$

$$y(t)=\frac{1}{2}x_1(t)-x_2(t)+\frac{1}{2}x_3(t)$$

The state model **in matrix form** is:

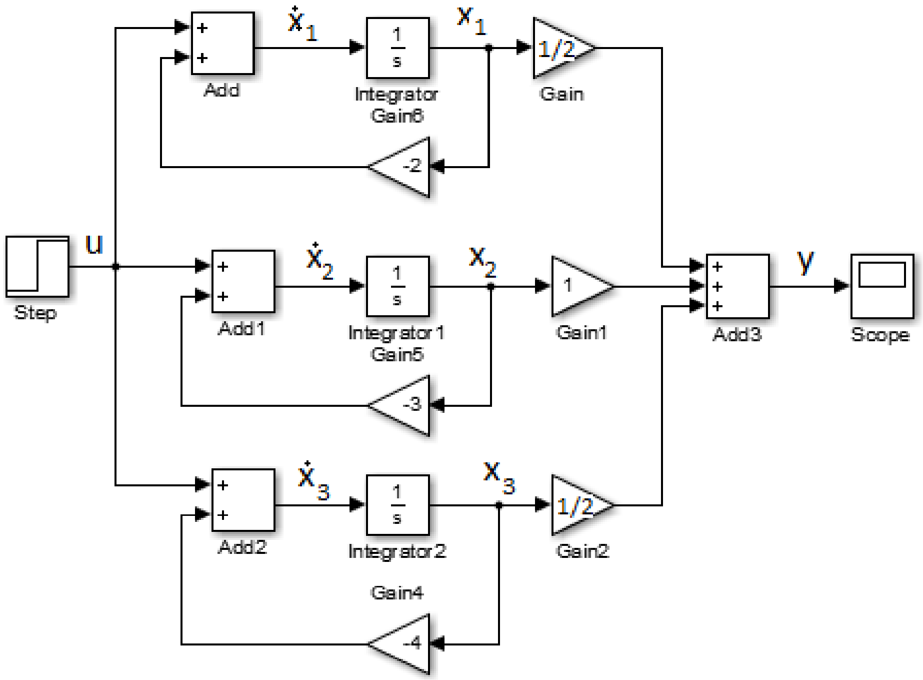
$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}-2 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}1/2 & -1 & 1/2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$A=\begin{bmatrix}-2 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -4\end{bmatrix} \quad B=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} \quad C=\begin{bmatrix}1/2 & -1 & 1/2\end{bmatrix}$$

Jordan's canonical form of state space description

Block diagram in Jordan form leads to the parallel connection of the basic elements:



Problem solved

Derive the state-space representation in **Frobenius** form for the system given by the transfer function.

$$G(p) = \frac{4p^2 + 3p + 1}{2p^2 + 3p + 5}$$

Solution for m = n:

1. Order of numerator decreasing: $G(p) = \dfrac{4p^2 + 3p + 1}{2p^2 + 3p + 5} = 2 + \dfrac{-3p - 9}{2p^2 + 3p + 5} = 2 + \dfrac{-3/2p - 9/2}{p^2 + 3/2p + 5/2}$

2. State equations:

$$\dot{x}_1(t) = x_2$$

$$\dot{x}_2(t) = -\frac{5}{2}x_1 - \frac{3}{2}x_2 + u$$

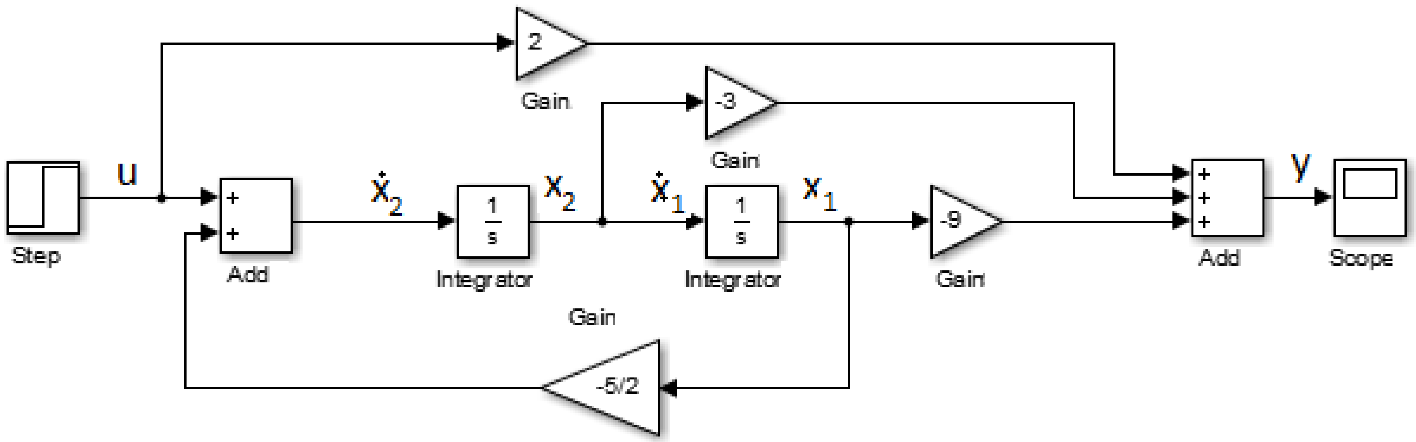
$$y(t) = -9x_1 - 3x_2 + 2u$$

3. State matrixes:

$$A = \begin{bmatrix} 0 & 1 \\ -5/2 & -3/2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} -9/2 & -3/2 \end{bmatrix} \qquad D = \begin{bmatrix} 2 \end{bmatrix}$$

Problem solved

4. State diagram:



5. Matrix *D*=[2] represents the direct connection from the input to the output.

Problems

1. Obtain state equations, state matrixes and state diagram in Frobenius form for differential equation.

$$\frac{d^2 y}{dt^2} + \frac{3\,dy}{dt} + 4y = \frac{du}{dt} + 3u$$

2. Obtain the state model in Jordan's form using parallel decomposition method.

$$G(p) = \frac{Y(p)}{U(p)} = \frac{p+3}{(p+1)\cdot(p+2)}$$

Problems

1. Derive the state transition matrix and transfer function from the state equations.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2. Derive the state transition matrix from the state equations.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 & 0{,}2 \\ 0 & 0{,}2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Revision questions

- Main differences between external and internal description of a system.
- Which forms of internal description do you know?
- Write the state-space description in general form.
- What is *x.*
- What is *A*.
- When the matrix D $\neq$ 0.
- How can we calculate poles of the system from matrix A.
- How we can calculate transfer function from matrix A,B,C,D.

References

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Individual Project

Individual project

INTERNAL DESCRIPTION OF SYSTEM

The system is given by differential equation:

$$y'' + 6\,y' + 8\,y = 3\,u' + 2\,u$$

- Describe the given system by the set of state-equations in Frobenius's and Jordan's form.
- Derive the state-matrixes A,B,C,D for both forms.
- Draw the state-block diagram for both forms in Matlab-Simulink and simulate the step responses.
- From state-matrixes A, B, C, D compute back the transfer function and verify in Matlab.

Individual project

EXTERNAL DESCRIPTION OF SYSTEM

The system is given by differential equation:

(1) $y'' + 7y' + 12y = 4u' + u$

(2) $y'' + 7y' + 10y = 2u' + 3u$

(3) $y'' + 8y' + 15y = 3u'$

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(12) $y'' + 10y' + 24y = u' + u$

(13) $y'' + 11y' + 30y = 2u' + u$

Individual project

EXTERNAL DESCRIPTION OF SYSTEM

The system is given by differential equation:

(14) $y'' + 11\,y' + 28\,y = 2\,u'$

(15) $y'' + 5\,y' + 4\,y = 2\,u'' + u$

(16) $y'' + 6\,y' + 5\,y = 4\,u'' + u$

(17) $y'' + 10\,y' + 24\,y = 3u'' + u$

(18) $y'' + 6\,y' + 8\,y = 5u'' + u$

(19) $y'' + 7\,y' + 12\,y = u'' + u$

(20) $y'' + 5\,y' + 6\,y = 2u''$

(21) $y'' + 7\,y' + 10\,y = 3\,u'' + u$

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Analysis of Single-loop Control System

Problem solved

Determine the control transfer function G_w, error transfer function G_e and disturbance transfer function G_z. Transfer functions of the system and controller are given.

$$G_s(p) = \frac{1}{(p+4)(p+1)} \qquad G_R(p) = \frac{3}{(p+3)}$$

$$G_w(p) = \frac{G_r(p)G_s(p)}{1+G_r(p)G_s(p)} = \frac{\frac{3}{(p+3)} \cdot \frac{1}{(p+4)\cdot(p+1)}}{1+\frac{3}{(p+3)} \cdot \frac{1}{(p+4)\cdot(p+1)}} = \frac{3}{(p+3)\cdot(p+4)\cdot(p+1)+3}$$

$$G_e(p) = \frac{1}{1+G_r(p)\,G_s(p)} = \frac{1}{1+\frac{3}{(p+3)} \cdot \frac{1}{(p+4)\cdot(p+1)}} = \frac{(p+1)\cdot(p+4)\cdot(p+1)}{(p+3)\cdot(p+4)\cdot(p+1)+3}$$

$$G_Z(p) = \frac{G_s(p)}{1+G_r(p)\,G_s(p)} = \frac{\frac{1}{(p+4)\cdot(p+1)}}{1+\frac{3}{(p+3)} \cdot \frac{1}{(p+4)\cdot(p+1)}} = \frac{(p+3)}{(p+3)\cdot(p+4)\cdot(p+1)+3}$$

Problem solved

Determine steady state value of

a) the output and

b) of the error signal.

$$G_s(p) = \frac{1}{(p+4)(p+1)} \qquad G_R(p) = \frac{3}{(p+3)}$$

Assume the reference input *w(t)*=1 and deviation input *z(t)*=1. Transfer functions of the system and controller are given. Apply final theorem of Laplace transform.

a1)

$$G_w(p) = \frac{Y(p)}{W(p)} = \frac{G_r(p)G_s(p)}{1+G_r(p)G_s(p)} \Rightarrow Y(p) = G_w(p)\,W(p)$$

$$y(\infty) = \lim_{p\to 0} p\cdot Y(p) = \lim_{p\to 0} p\cdot \frac{3}{(p+3)\cdot(p+4)\cdot(p+1)+3}\cdot\frac{1}{p} = \frac{1}{5}$$

a2)

$$G_Z(p) = \frac{Y(p)}{Z(p)} = \frac{G_s(p)}{1+G_r(p)G_s(p)} \Rightarrow Y(p) = G_Z(p)\,Z(p)$$

$$y(\infty) = \lim_{p\to 0} p\cdot Y(p) = \lim_{p\to 0} p\cdot \frac{p+3}{(p+3)\cdot(p+4)\cdot(p+1)+3}\cdot\frac{1}{p} = \frac{1}{5}$$

b)

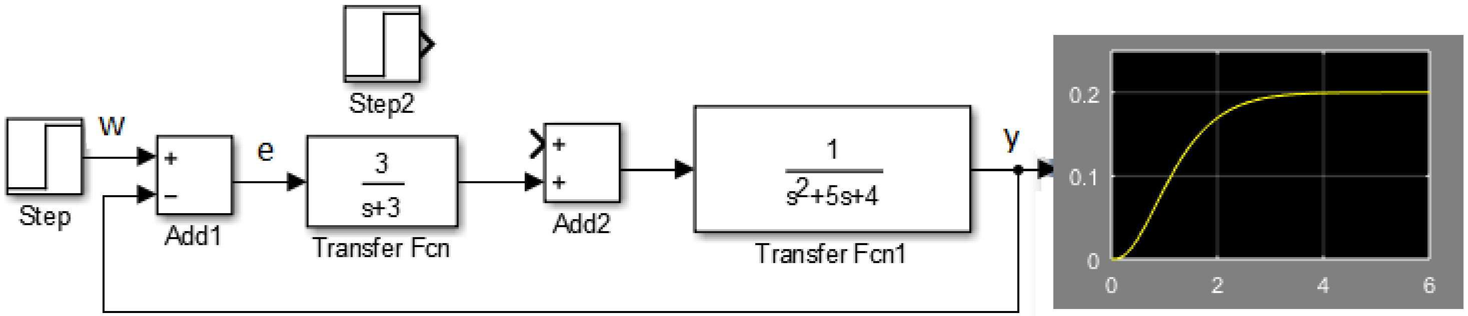
$$G_e(p) = \frac{E(p)}{W(p)} = \frac{1}{1+G_r(p)G_s(p)} \Rightarrow E(p) = G_e(p)\,W(p)$$

$$e(\infty) = \lim_{p\to 0} p\cdot E(p) = \lim_{p\to 0} p\cdot \frac{(p+3)\cdot(p+4)\cdot(p+1)}{(p+3)\cdot(p+4)\cdot(p+1)+3}\cdot\frac{1}{p} = \frac{4}{5}$$

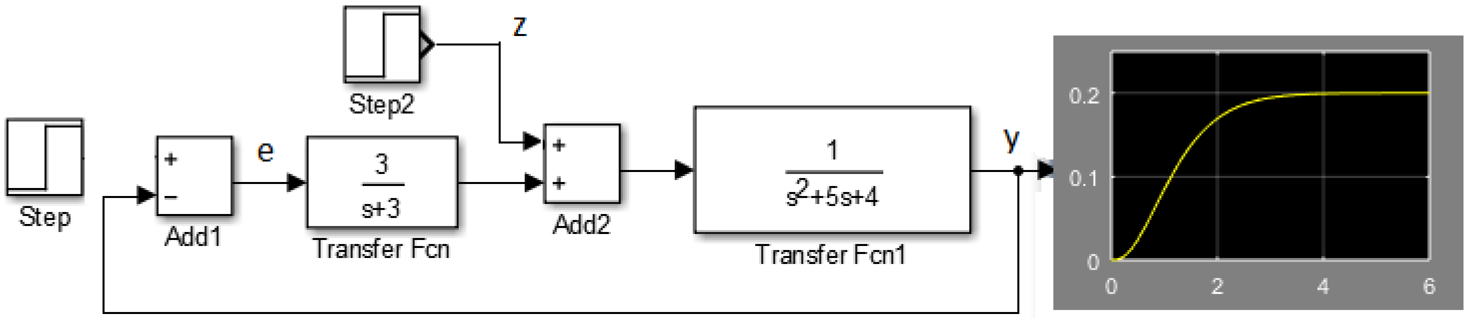
Problem solved

Simulation of step responses in Matlab.

a1) step response – steady state value of output if *w(t)*=1



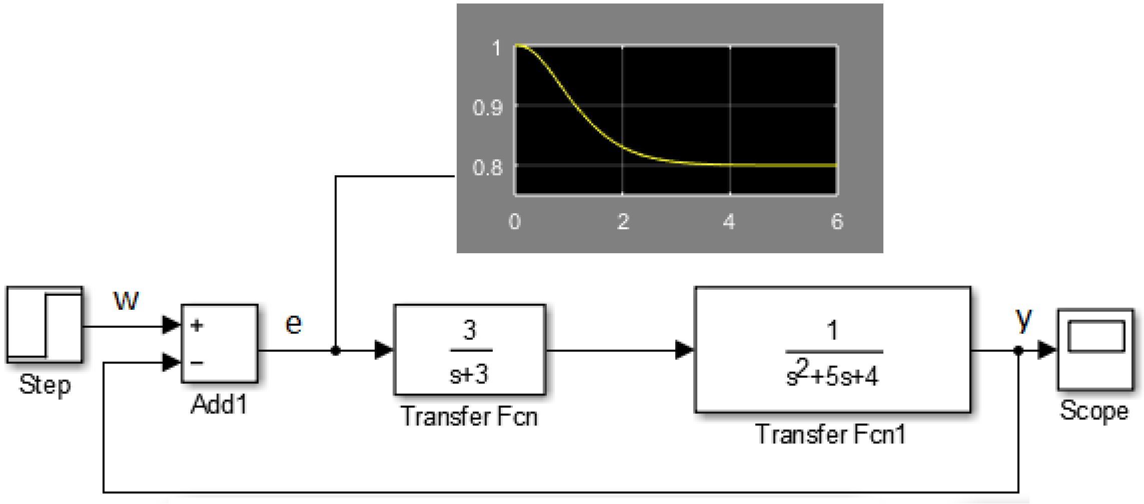
a2) step response – steady state value of output if *z(t)*=1



Problem solved

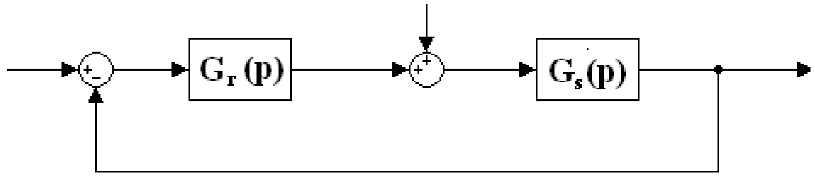
Simulation of the step response in Matlab.

b) Steady state value of error signal, if *w(t)*=1



Revision

- Name the main signals in the following control system.



- Explain the principle of this control system.
- <u>Write the formulae of:</u>
  - Control transfer function
  - Deviation (Error) transfer function
  - Disturbance transfer function
- Write the formula for the calculation of steady state error if the input is the unit step.
- Write the formula for the calculation of steady state output if the input is the unit step.
- <u>Write the transfer functions of:</u>
  - P-controller
  - I-controller
  - D-controller. (Is it possible to use D-controller by itself?)
  - PI-controller
  - Ideal PD-controller, real PD-controller
  - Ideal PID-controller, real PID-controller
- Describe general features of P, I, D, PI, PD, PID controllers.
- How do the controllers influence the control process?

References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.
ISBN: 81-7409-107-6

[2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing

[3] Matlab&Simulink: R2015a

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9. Stability Analysis of Closed-loop Control System

Necessary condition of stability: (but not sufficient)

Using necessary condition of stability determin stability of the systems done by characteristic polynomials

- $A(p) = p^6 + 7p^4 + 17p^2 + 9$
- $A(p) = p^3 + 2p^2 + 9p + 68$
- $A(p) = p^4 + 2p^3 - 4p^2 + 10p - 1$
- $A(p) = 4p^2 + 8p + 4$
- $A(p) = -4p^2 - 8p - 4$

Problems

Using Hurwitz or Routh-Schur criterion determine the stability of the systems done by characteristic polynomial.

- $A(p) = p^3 + 2p^2 + 9p + 68$
- $A(p) = 2p^4 + p^3 + 5p^2 + 3p + 4$
- $A(p) = 2p^4 + 5p^3 + 5p^2 + 2p + 1$
- $A(p) = - p^5 - 3p^4 - 10p^3 - 12p^2 - 7p - 3$

Confirm results using the Matlab function “roots”.

Problem solved

Find the ultimate gain r_0 for stable closed-loop system. If we have only a proportional controller $G_r = r_0$ and system given by the transfer function:

$$G_s = \frac{1}{(p+3)(p+2)(p+1)}$$

Control transfer function:

$$G_w(p) = \frac{Y(p)}{W(p)} = \frac{G_r G_s}{1+G_r G_s} = \frac{\frac{r_0}{(p+3)(p+2)(p+1)}}{1+\frac{r_0}{(p+3)(p+2)(p+1)}} = \frac{r_0}{(p+3)(p+2)(p+1)+r_0} = \frac{r_0}{p^3+6p^2+11p+(6+r_0)}$$

Characteristic polynomial: $A(p) = p^3 + 6p^2 + 11p + (6+r_0)$

Hurwitz determinant:

$$H = \begin{vmatrix} 6 & 6+r_0 & 0 \\ 1 & 11 & 0 \\ 0 & 6 & 6+r_0 \end{vmatrix}$$

Hurwitz subdeterminants:

$$H_1 = 6 > 0$$

$$H_2 = \begin{vmatrix} 6 & 6+r_0 \\ 1 & 11 \end{vmatrix} = 66 - 6 - r_0 = 60 - r_0 > 0$$

$$H_3 = H_2(6+r_0) > 0$$

$$\boxed{r_0 < 60 \quad and \quad r_0 > -6 \quad \rightarrow \quad r_0 \in (-6; 60)}$$

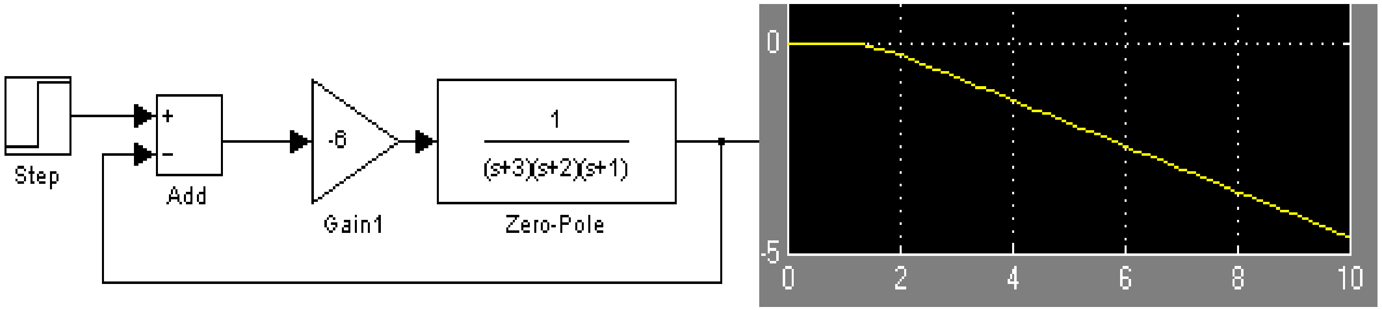
Simulation in Matlab

a) Poles: 0.0000 + 0.0000i

-3.0000 + 1.4142i

-3.0000 - 1.4142i

System unstable.



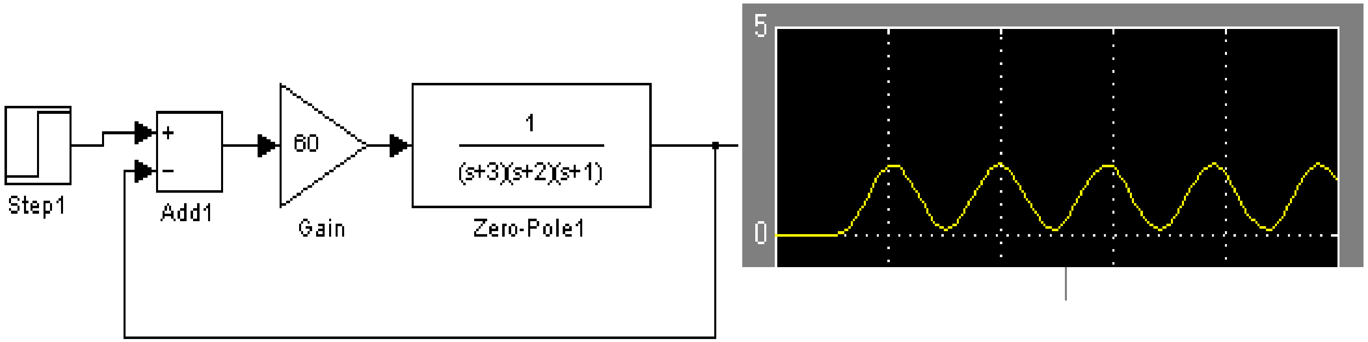
b) Poles: -6.0000 + 0.0000i

-0.0000 + 3.3166i

-0.0000 - 3.3166i

System on stability limit:

sustained oscillation



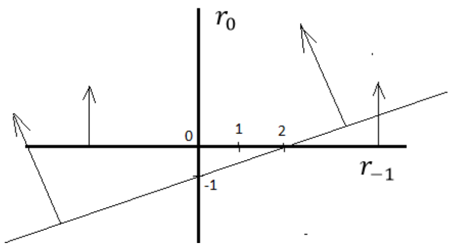
Problem solved

Assume the second-order transfer function. If we synthesize the control with PI controller, what are the stability constraints?

$$G_s = \frac{1}{p^2 + 2p + 1} \qquad G_r = r_0 + \frac{r_{-1}}{p} = \frac{r_0 p + r_{-1}}{p}$$

$$G_w = \frac{Y(p)}{W(p)} = \frac{G_r \cdot G_s}{1 + G_r \cdot G_s} = \frac{\dfrac{r_0 p + r_{-1}}{p} \dfrac{1}{p^2 + 2p + 1}}{1 + \dfrac{r_0 p + r_{-1}}{p\left(p^2 + 2p + 1\right)}} = \frac{r_0 p + r_{-1}}{p^3 + 2p^2 + (1 + r_0) p + r_{-1}}$$

The characteristic equation is: $p^3 + 2p^2 + (1 + r_0)p + r_{-1} = 0$



$$H = \begin{vmatrix} 2 & r_{-1} & 0 \\ 1 & 1 + r_0 & 0 \\ 0 & 2 & r_{-1} \end{vmatrix}$$

$$H_1 = 2 > 0$$

$$H_2 = 2(1 + r_0) + r_{-1} > 0 \quad \Rightarrow \quad 2r_0 > r_{-1} - 2 \Rightarrow \quad r_0 > \frac{1}{2} r_{-1} - 1$$

$$\text{or} \qquad r_{-1} < 2r_0 + 2$$

Problems

1. Find the stability constraints.

$$G_s = \frac{1}{a_2\, p^2 + a_1\, p} \qquad\qquad a_2, a_1 \rangle\, 0$$

$$a) \quad G_R = \frac{r_{-1}}{p}$$

$$b) \quad G_R = \quad r_0 + \frac{r_{-1}}{p}$$

2. Characteristic equation is: *A(p) = 0,002p³ + 0,08p² + p(0,15K – 1) + K*

Find the condition for *K* for the system to be unstable.

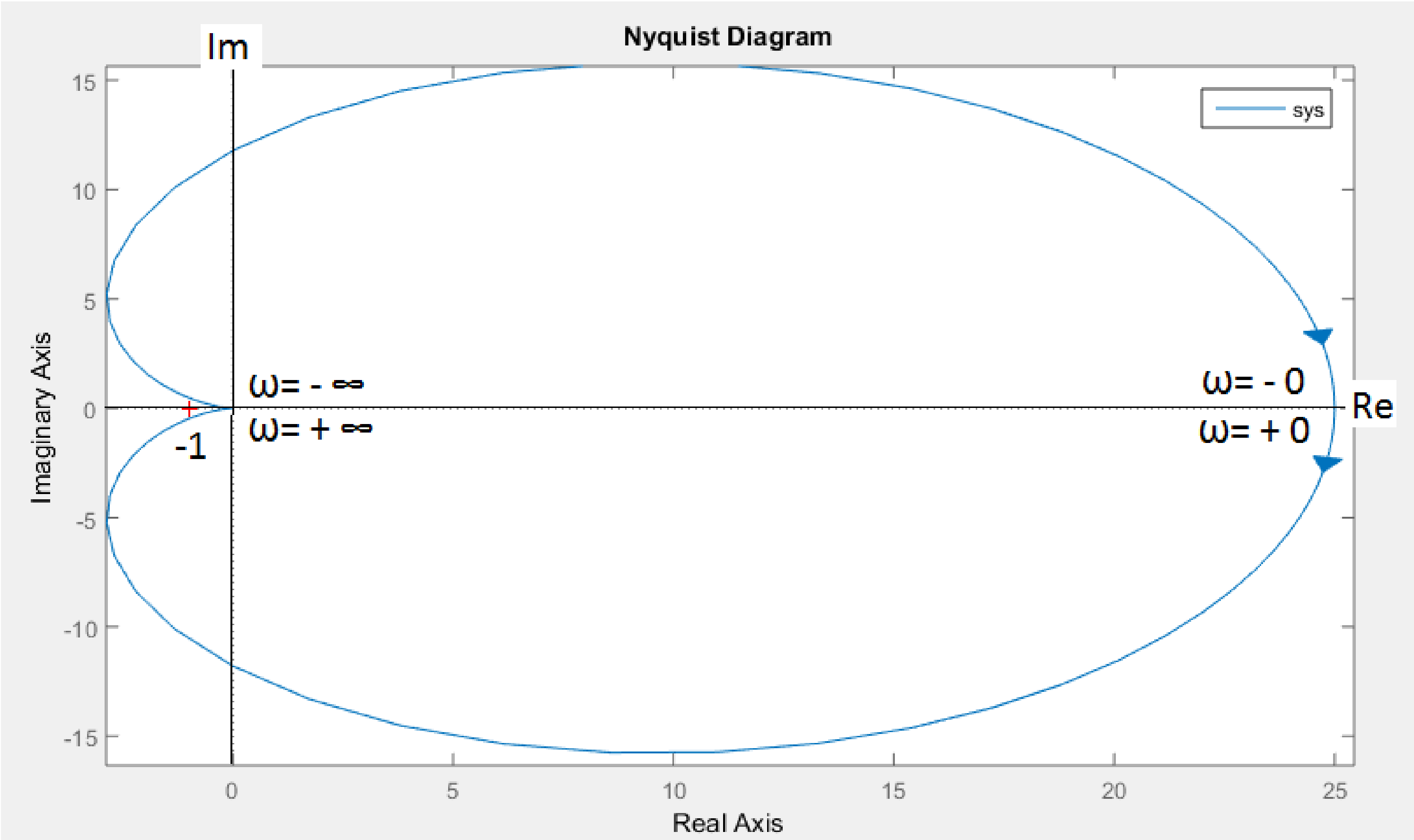
Solved example: Nyquist criterion

Examine the closed-loop stability of the system whose open-loop transfer function is given by:

$$G_0(p) = \frac{50}{(p+1)(p+2)}$$

Using Matlab:

```
num =
     50
>> den=[1 3 2]
den =
     1    3    2
>> nyquist(num,den)
```



Revision questions

- Which of components of PID:

  a) can accelerate the response of the control system

  b) can eliminate the error-steady state value

  c) can extend the settling time

- Where are the poles of the transfer function located, if
  - the system is stable
  - the system is unstable
  - the system is oscillating with constant amplitude

- The necessary condition of stability.
- Which algebraic criteria of stability do you know?

References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.
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[2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing

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10. Quality of Control Performance.

Solved example

Determine steady state deviation. Input signal is W(p).

1. $\boxed{\text{s = 0, r = 0:}}$ $G_s = \frac{1}{p^2+2p+1}$ $G_R = r_0$ $W(p) = \frac{w_0}{p}$

$$G_e = \frac{E(p)}{W(p)} = \frac{1}{1+G_s G_r} = \frac{1}{1+\frac{r_0}{p^2+2p+1}} = \frac{p^2+2p+1}{p^2+2p+(1+r_0)}$$

$$E(p) = G_e \frac{w_0}{p}$$

$$e(\infty) = \lim_{p\to 0} p \cdot G_e \cdot \frac{w_0}{p} = \frac{w_0}{1+r_0}$$

Solved example

Determine steady state deviation. Input signal is W(p).

2. $\boxed{s = 0, r = 1:}$ $G_s = \frac{1}{p^2 + 2p + 1}$ $G_R = \frac{r_{-1}}{p}$ $W(p) = \frac{w_0}{p}$

$$G_e = \frac{E(p)}{W(p)} = \frac{1}{1 + G_s G_r} = \frac{1}{1 + \frac{1}{p^2 + 2p + 1} \cdot \frac{r_{-1}}{p}} = \frac{p \cdot (p^2 + 2p + 1)}{p^3 + 2p^2 + p + r_{-1})}$$

$$E(p) = G_e \frac{w_0}{p}$$

$$e(\infty) = \lim_{p \to 0} p \cdot G_e \cdot \frac{w_0}{p} = \lim_{p \to 0} p \cdot \frac{p \cdot (p^2 + 2p + 1)}{p^3 + 2p^2 + p + r_{-1})} \cdot \frac{w_0}{p} = 0$$

Solved example

Determine steady state deviation. Input signal is W(p).

3. $\boxed{s = 0, r = 1:}$ $G_s = \frac{1}{p^2+2p+1}$ $G_R = r_0 + \frac{r_{-1}}{p}$ $W(p) = \frac{w_0}{p}$

$$G_e = \frac{E(p)}{W(p)} = \frac{1}{1+G_s G_r} = \frac{1}{1+\frac{1}{p^2+2p+1}\cdot\frac{r_0 p + r_{-1}}{p}} = \frac{p\cdot(p^2+2p+1)}{p^3+2p^2+p(1+r_0)+r_{-1}}$$

$$E(p) = G_e \frac{w_0}{p}$$

$$e(\infty) = \lim_{p\to 0} p \cdot G_e \cdot \frac{w_0}{p} = \lim_{p\to 0} p \cdot \frac{p\cdot\left(p^2+2p+1\right)}{p^3+2p^2+p(1+r_0)+r_{-1}} \cdot \frac{w_0}{p} = 0$$

Solved example

Determine steady state deviation. Input signal is W(p).

4. $\boxed{\text{s = 0, r = 1:}}$ $\quad G_s = \frac{1}{p^2+2p+1} \quad G_R = \frac{r_{-1}}{p} \quad W(p) = \frac{w_1}{p^2}$

$$G_e = \frac{E(p)}{W(p)} = \frac{1}{1+G_s G_r} = \frac{1}{1+\frac{1}{p^2+2p+1}\cdot\frac{r_{-1}}{p}} = \frac{p\cdot(p^2+2p+1)}{p^3+2p^2+p+r_{-1}}$$

$$E(p) = G_e \frac{w_1}{p^2}$$

$$e(\infty) = \lim_{p\to 0} p\cdot G_e \cdot \frac{w_1}{p^2} = \lim_{p\to 0} p\cdot\frac{p\cdot(p^2+2p+1)}{p^3+2p^2+p+r_{-1}}\cdot\frac{w_1}{p^2} = \frac{w_1}{r_0}$$

Solved example

Determine steady state deviation. Input signal is disturbance Z(p).

1. s = 0, r = 0: $\quad G_s = \dfrac{1}{p^2 + 2p + 1} \qquad G_R = r_0 \qquad Z(p) = \dfrac{z_0}{p}$

$$G_z = \frac{Y(p)}{Z(p)} = \frac{G_s}{1 + G_s G_r} = \frac{\dfrac{1}{p^2 + 2p + 1}}{1 + \dfrac{1}{p^2 + 2p + 1} \cdot r_0} = \frac{1}{p^2 + 2p + 1 + r_0}$$

$$Y(p) = G_z \cdot \frac{z_0}{p}$$

$$(\infty) = -y(\infty) = -\lim_{p \to 0} p \cdot G_z \cdot \frac{z_0}{p} == -\lim_{p \to 0} p \cdot \frac{1}{p^2 + 2p + 1 + r_0} \cdot \frac{z_0}{p} = -\frac{z_0}{1 + r_0}$$

Solved example

Determine steady state deviation. Input signal is disturbance Z(p).

2. $\boxed{\text{s= 0, r = 0:}}$ $G_s = \frac{1}{p^2+2p+1}$ $G_R = r_0$ $Z(p) = \frac{z_1}{p^2}$

$$G_z = \frac{Y(p)}{Z(p)} = \frac{G_s}{1+G_sG_r} = \frac{\frac{1}{p^2+2p+1}}{1+\frac{1}{p^2+2p+1}\cdot r_0} = \frac{1}{p^2+2p+1+r_0}$$

$$Y(p) = G_z \cdot \frac{z_1}{p^2}$$

$$(\infty) = -y(\infty) = -\lim_{p\to 0} p \cdot G_z \cdot \frac{z_1}{p^2} == -\lim_{p\to 0} p \cdot \frac{1}{p^2+2p+1+r_0} \cdot \frac{z_1}{p^2} = -\infty$$

Solved example

Determine steady state deviation. Input signal is disturbance Z(p).

3. $\boxed{s = 1,\ r = 0:}$

$$G_s = \frac{1}{p\cdot(p+1)} \qquad G_R = r_0 \qquad Z(p) = \frac{z_0}{p}$$

$$G_z = \frac{Y(p)}{Z(p)} = \frac{G_s}{1 + G_s G_r} = \frac{\frac{1}{p\cdot(p+1)}}{1 + \frac{1}{p\cdot(p+1)}\cdot r_0} = \frac{1}{p^2 + p + r_0}$$

$$Y(p) = G_z \cdot \frac{z_0}{p}$$

$$(\infty) = -y(\infty) = -\lim_{p\to 0} p \cdot G_z \cdot \frac{z_0}{p} == -\lim_{p\to 0} p \cdot \frac{1}{p^2 + p + r_0}\cdot\frac{z_0}{p} = -\frac{z_0}{r_0}$$

Solved example

Determine steady state deviation. Input signal is disturbance Z(p).

4. $\boxed{\text{s= 0, r = 1:}}$

$$G_s = \frac{1}{p^2 + 2p + 1} \qquad G_R = r_0 + \frac{r_{-1}}{p} \qquad Z(p) = \frac{z_0}{p}$$

$$G_z = \frac{Y(p)}{Z(p)} = \frac{G_s}{1 + G_s G_r} = \frac{\frac{1}{p^2 + 2\mathrm{p} + 1}}{1 + \frac{1}{p^2 + 2\mathrm{p} + 1} \cdot \frac{r_0 p + r_{-1}}{p}} = \frac{\mathrm{p}}{p^3 + 2p^2 + p(1 + r_0) r_{-1}}$$

$$Y(p) = G_z \cdot \frac{z_0}{p}$$

$$(\infty) = -y(\infty) = -\lim_{p \to 0} p \cdot G_z \cdot \frac{z_0}{p} == -\lim_{p \to 0} p \cdot \frac{\mathrm{p}}{p^3 + 2p^2 + p(1 + r_0) r_{-1}} \cdot \frac{z_0}{p} = 0$$

Solved example

Determine steady state deviation. Input signal is disturbance Z(p).

5. s= 0, r = 1:

$$G_s = \frac{1}{p^2 + 2p + 1} \qquad G_R = r_0 + \frac{r_{-1}}{p} \qquad Z(p) = \frac{z_1}{p^2}$$

$$G_z = \frac{Y(p)}{Z(p)} = \frac{G_s}{1 + G_s G_r} = \frac{\frac{1}{p^2 + 2\mathrm{p} + 1}}{1 + \frac{1}{p^2 + 2\mathrm{p} + 1} \cdot \frac{r_0 p + r_{-1}}{p}} = \frac{\mathrm{p}}{p^3 + 2p^2 + p(1 + r_0) + r_{-1}}$$

$$Y(p) = G_z \cdot \frac{z_1}{p^2}$$

$$(\infty) = -y(\infty) = -\lim_{p \to 0} p \cdot G_z \cdot \frac{z_1}{p^2} == -\lim_{p \to 0} p \cdot \frac{\mathrm{p}}{p^3 + 2p^2 + p(1 + r_0) + r_{-1}} \cdot \frac{z_1}{p^2} = -\frac{z_1}{r_{-1}}$$

Solved example

Determine steady state deviation. Input signal is disturbance Z(p).

6. s= 1, r = 1: $G_s = \dfrac{1}{p(p+1)}$ $G_R = r_0 + \dfrac{r_{-1}}{p}$ $Z(p) = \dfrac{z_0}{p}$

$$G_z = \frac{Y(p)}{Z(p)} = \frac{G_s}{1 + G_s G_r} = \frac{\frac{1}{p \cdot (p+1)}}{1 + \frac{1}{p \cdot (p+1)} \cdot \frac{r_0 p + r_{-1}}{p}} = \frac{\mathrm{p}}{p^2 + p(1 + r_0) + r_{-1}}$$

$$Y(p) = G_z \cdot \frac{z_0}{p}$$

$$(\infty) = -y(\infty) = -\lim_{p \to 0} p \cdot G_z \cdot \frac{z_0}{p} = -\lim_{p \to 0} p \cdot \frac{\mathrm{p}}{p^2 + p(1 + r_0) + r_{-1}} \cdot \frac{z_0}{p} = 0$$

Solved example

Determine steady state deviation. Input signal is disturbance Z(p).

7. s= 1, r = 1: $\qquad G_s = \dfrac{1}{p(p+1)} \qquad G_R = r_0 + \dfrac{r_{-1}}{p} \qquad Z(p) = \dfrac{z_1}{p^2}$

$$G_z = \frac{Y(p)}{Z(p)} = \frac{G_s}{1 + G_s G_r} = \frac{\dfrac{1}{p\cdot(p+1)}}{1 + \dfrac{1}{p\cdot(p+1)}\cdot\dfrac{r_0 p + r_{-1}}{p}} = \frac{\mathrm{p}}{p^2 + p(1 + r_0) + r_{-1}}$$

$$Y(p) = G_z \cdot \frac{z_1}{p^2}$$

$$(\infty) = -y(\infty) = -\lim_{p\to 0} p \cdot G_z \cdot \frac{z_1}{p^2} = -\lim_{p\to 0} p \cdot \frac{\mathrm{p}}{p^2 + p(1 + r_0) + r_{-1}} \cdot \frac{z_1}{p^2} = -\frac{z_1}{r_{-1}}$$

Aperiodicity criterion: Problem solved

Characteristic polynomial is $A(p) = 4p^2 + kp + 2$

Find k to set the step response on the boundary of aperiodicity.

4	8	k	k	2	$\alpha_0 = \frac{4}{8} = \frac{1}{2}$
8	0	k	0	0	
$4 - \frac{1}{2}8$	8	$k - \frac{k}{2}$	k	2	$\alpha_1 = \frac{8}{\frac{k}{2}} = \frac{16}{k}$
	$\frac{k}{2}$	0	2	0	
	0	$\frac{k}{2}$	$k - \frac{32}{k}$	2	

Condition for boundary of aperiodicity: $$\boldsymbol{k - \frac{32}{k} = 0}$$

1) $\boxed{\boldsymbol{k_1 = +\sqrt{32}}}$

2) $\boldsymbol{k_2 = -\sqrt{32}}$

Problem

Assume the system given by the transfer function: $G_s = \dfrac{1}{(10p+1)^2}$

Three controllers were designed for this system. Compute the quadratic control area and decide which controller is the best according to the quadratic integral criterion. Simulate and compare the step responses in Matlab.

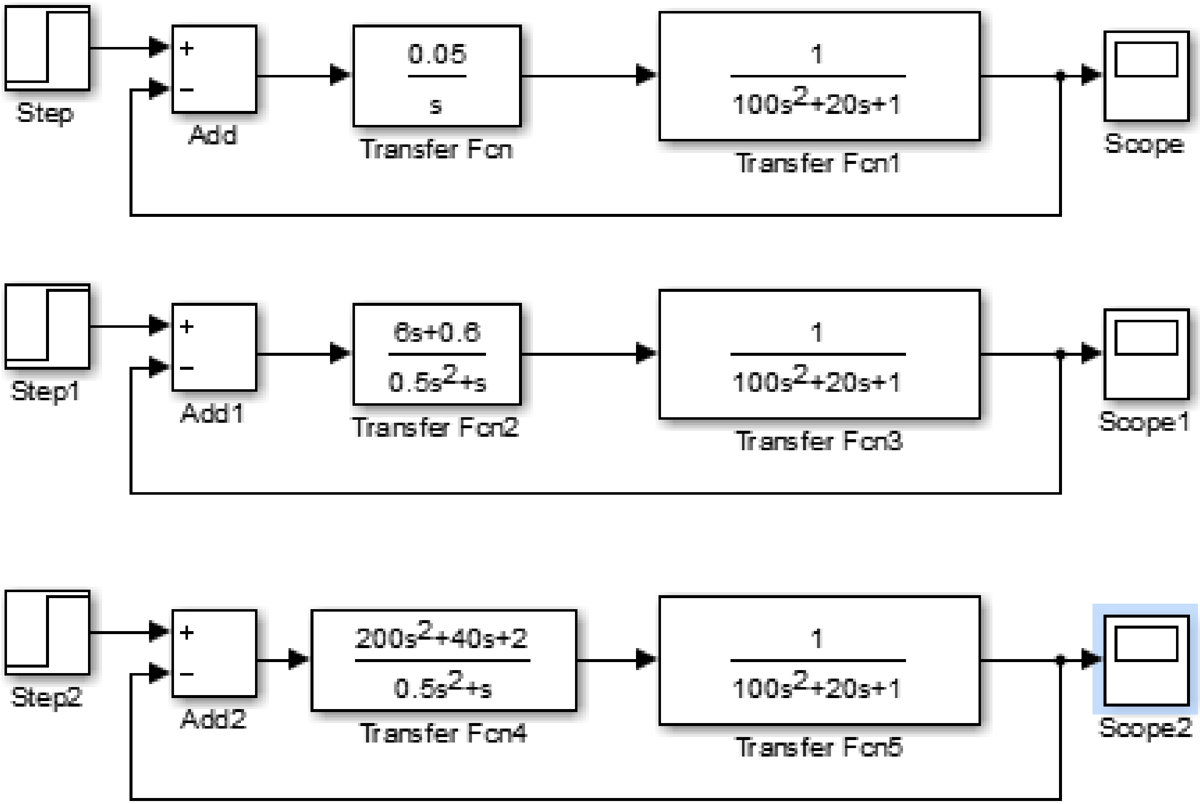
$$G_{R1} = \frac{0{,}05}{p}$$

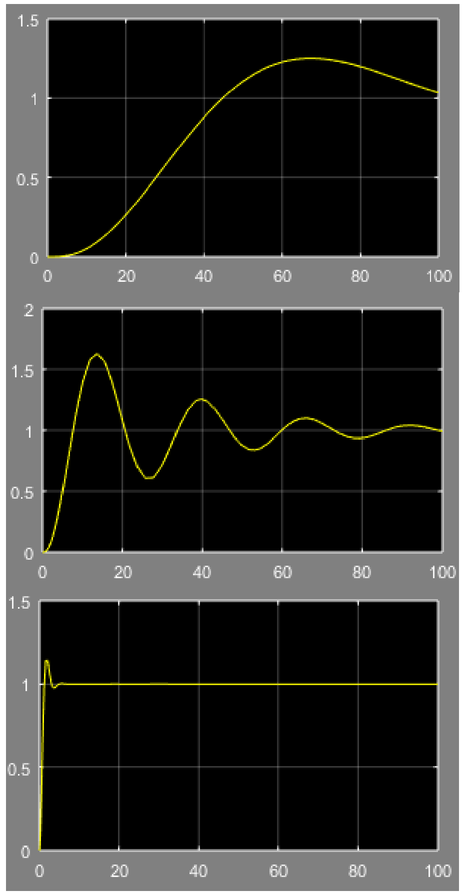
$$G_{R2} = \frac{0{,}6(10p+1)}{p(0{,}5p+1)}$$

$$G_{R3} = \frac{2(10p+1)^2}{p(0{,}5p+1)}$$

Results: 1) Iq = 23,3 2) Iq = 8,16 3) Iq=0,5

Solution: Simulation in Matlab





References

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11. Design of Controllers

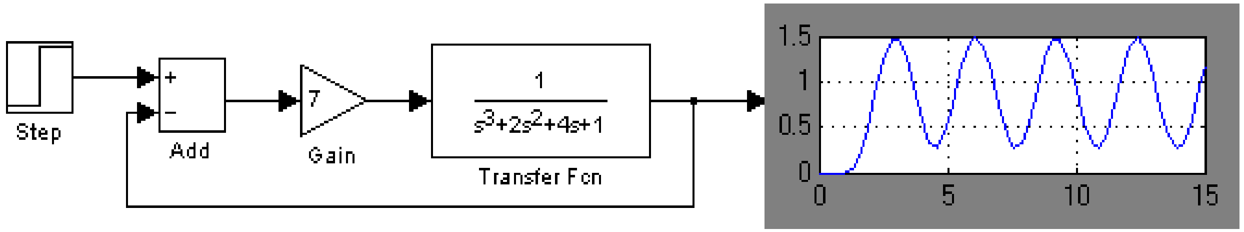
Ziegler-Nichols: Illustrative example

Design the P-controller for the system given by transfer function using the Ziegler-Nichols method.

$$G_s = \frac{1}{p^3 + 2p^2 + 4p + 1}$$

$$G_R = r_0 = ?$$

a) Simulation in Matlab:

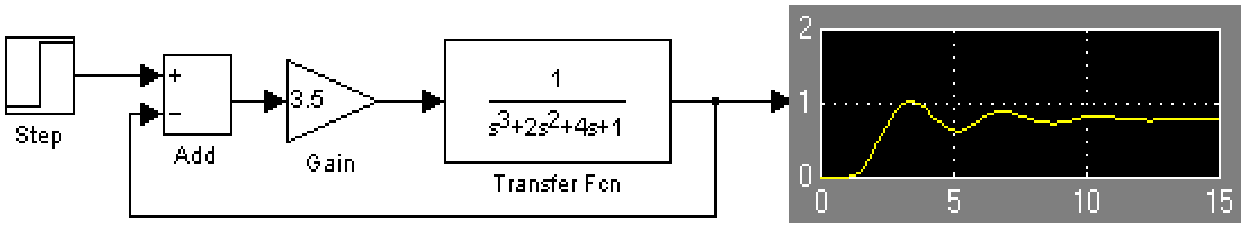


$R_{0k} = 7$

$T_k = 3$

b) Optimal setting of P controller: $r_0 = 0{,}5.r_{0k} = 0{,}5.7 = 3{,}5$

Simulation in Matlab:



Ultimate gain r_{0k} using substitution p = jω

Closed-loop transfer function:

$$G_w = \frac{G_R G_S}{1+G_R G_S} = \frac{\dfrac{r_0}{p^3+2p^2+4p+1}}{1+\dfrac{r_0}{p^3+2p^2+4p+1}} = \frac{r_0}{p^3+2p^2+4p+1+r_0}$$

Characteristic polynomial on stability limit: $A(p) = p^3 + 2p^2 + 4p + 1 + r_0 = 0$

After substitution: $A(j\omega) = (j\omega)^3 + 2(j\omega)^2 + 4j\omega + 1 + r_0 = -j\omega^3 - 2\omega^2 + 4j\omega + 1 + r_0 = 0 \quad \Leftrightarrow$

$$Re\{A(j\omega)\} = 0 \;\; and \;\; Im\{A(j\omega)\} = 0$$

Im: $\quad -\omega^3 + 4\omega = 0 \;\Rightarrow\; -\omega(\omega^2 - 4) = 0 \;\Leftrightarrow\; \omega_1 = 0 \;\; and \;\; \omega_{1,2} = \pm 2$

Re: $\quad 2\omega^2 + 1 + r_0 = 0 \;\Rightarrow\; for \;\; \omega = \pm 2: \;\; -8 + 1 + r_0 = 0 \;\Leftrightarrow\; r_0 = 7$

$\qquad for \;\; \omega = 0: \;\; 1 + r_0 = 0 \;\Leftrightarrow\; r_0 = -1$

For r_0 = 7 the system exhibits sustained oscillation, therefore the system is on the verge of instability or marginally stable.

Multiple roots optimum adjustment

Guarantee the aperiodic response.

For transfer function: $G_w = \frac{c^n}{(p+c)^n}$

where *c* is multiple root. Numerators' form is not obligatory, only recommended for optimum adjustment.

- for second-order system: $A(p) = p^2 + 2\,p\,c + c^2$
- for third-order system: $A(p) = p^3 + 3\,p^2\,c + 3\,c^2\,p + c^3$
- for forth-order system: $A(p) = p^4 + 4\,p^3\,c + 6\,c^2\,p^2 + 4\,c^3\,p + c^4$

Standard forms: Example solved

Design a PI controller for the system given by the transfer function: $G_S = \frac{1}{3p^2 + 2p}$

$$G_R = r_0 + \frac{r_{-1}}{p}$$

$$G_w = \frac{G_R G_S}{1 + G_R G_S} = \frac{r_0 p + r_{-1}}{3p^3 + 2p^2 + r_0 p + r_{-1}} = \frac{\frac{1}{3}(r_0 p + r_{-1})}{p^3 + \frac{2}{3}p^2 + \frac{r_0}{3}p + \frac{r_{-1}}{3}}$$

a) ITAE

Characteristic polynomial: $A(p) = p^3 + \frac{2}{3}p^2 + \frac{r_0}{3}p + \frac{r_{-1}}{3}$

Standard form of characteristic polynomial: $A(p) = p^3 + 2\,\omega\, p^2 + 19{,}2\,\omega^2\, p + \omega^3$

Compare coefficients:

- at p^2: $2/3 = 3\,c$ → $c = 2/9$
- at p^1: $r_0/3 = 3\,c^2$ → $r_0 = 4/9$
- at p^0: $r_{-1}/3 = c^3$ → $r_{-1} = 8/243$

$$G_r = 6{,}4 + \frac{1/9}{p}$$

b) Multiple root adjustment

The characteristic polynomial is: $$A(p) = p^3 + \frac{2}{3}p^2 + \frac{r_0}{3}p + \frac{r_{-1}}{3}$$

We compare it with the standard form: $A(p) = p^3 + 3\,c\,p^2 + 3\,c^2\,p + c^3$

- p^2: $2/3 = 3\,c$ → $c = 2/9$
- p^1: $r_0/3 = 3\,c^2$ → $r_0 = 4/9$
- p^0: $r_{-1}/3 = c^3$ → $r_{-1} = 8/243$

$$G_r = \frac{4}{9} + \frac{8/243}{p}$$

c) **Dominant roots adjustment**

The characteristic polynomial is: $$A(p) = p^3 + \frac{2}{3}p^2 + \frac{r_0}{3}p + \frac{r_{-1}}{3}$$

We compare it with standard form: $A(p) = p^3 + 7\,c\,p^2 + 12\,c^2\,p + 10\,c^3$

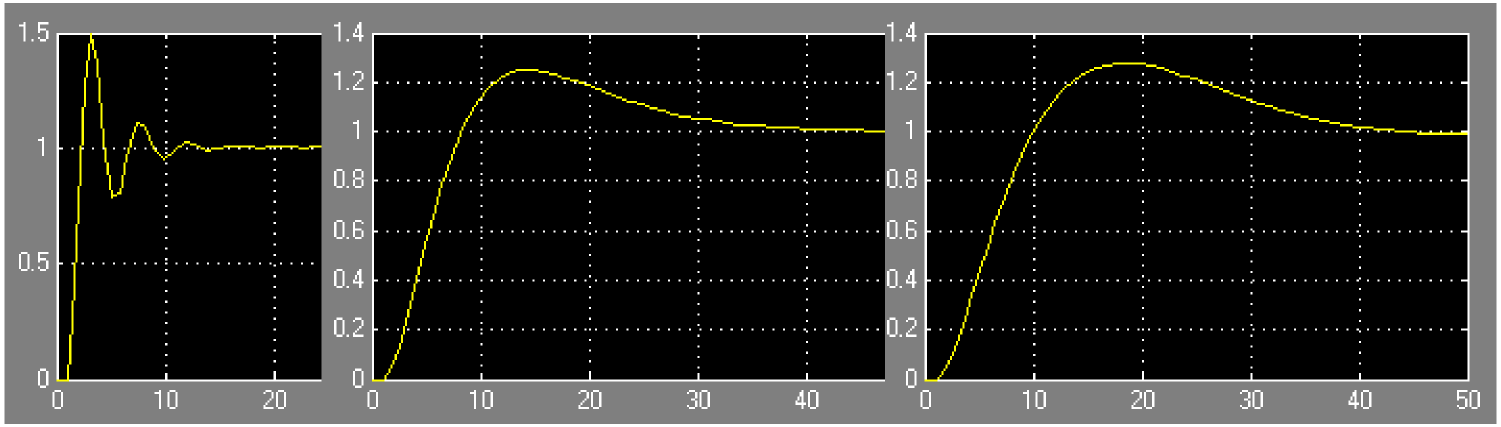
- p^2: $2/3 = 7\,c$ → $c = 2/21$
- p^1: $r_0/3 = 12\,c^2$ → $r_0 = 16/49$
- p^0: $r_{-1}/3 = 10\,c^3$ → $r_{-1} = 80/3177$

$$G_r = \frac{16}{49} + \frac{80/3177}{p}$$

Simulation in Matlab

Step responses for:

a) ITAE b) Multiple roots adjustment c) Dominant roots adjustment



Solved example

$$G_S = \frac{1}{p^3 + 3p^2 + 2p} \qquad G_R = r_0$$

$$G_w = \frac{G_R G_S}{1 + G_R G_S} = \frac{r_0}{p^3 + 3p^2 + 2p + r_0}$$

- **ITAE**

Characteristic polynomial is: $A(p) = p^3 + 3p^2 + 2p + r_0$

We compare it with standard form: $A(p) = p^3 + 1{,}72\,\omega\, p^2 + 2{,}17\,\omega^2 p + \omega^3$

Coefficients of the same powers must be equal, but in this case we obtain two equations for ω and the set of equations cannot be solved.

- p^2: $3 = 1{,}72\,\omega$
- p^1: $2 = 2{,}17\,\omega^2$
- p^0: $r_0 = \omega^3$

We cannot even use any other standard form.

Problem

Design optimal controller for the system and the input signal given bellow.

$$G_S = \frac{1}{8p^2 + 22p + 12} \qquad w(t) = k \cdot t$$

Individual project

1. System is given by transfer function $G_s = \dfrac{1}{(12p+1)(3p+1)}$; assume disturbance z(t) = 1.

Adjust the parameters of I and PI controllers using

a) Ziegler-Nichols method

b) Optimum module criterion.

c) Compare control quality using quadratic integral criterion

d) Simulate all responses in Matlab-Simulink

2. System is given in state-space by matrixes

$$A = \begin{bmatrix} -1 & 2 & -1 \\ 1 & -3 & 1 \\ 1 & 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

a) Calculate the poles of the system

b) Calculate the controllability and observability matrix and decide if the model is controllable and observable.

c) Find the state feedback gain K to obtain Tmax (time of first overshoot). Tmax = 2 ; k = 5

References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.
ISBN: 81-7409-107-6

[2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing

[3] Matlab&Simulink: R2015a

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12. Standard Methods of Control Adjustment

Quadratic control area: Problems

Determine I-controller using quadratic integral criterion:

1) $G_S = \dfrac{1}{2{,}5p^3 + 8p^2 + 6{,}5p + 1}$ $\quad G_R = \dfrac{r_{-1}}{p}$ $\quad w(t) = 1$

2) $G_S(p) = \dfrac{1}{2p^2 + 6p + 4}$ $\quad G_R = \dfrac{r_{-1}}{p}$ $\quad z(t) = 1$

1) Solution: $r_{-1} = -0{,}6529$ (unstable solution)

$\boxed{r_{-1} = 0{,}2432}$ (optimal setting)

2) Solution: $r_{-1} = 6$

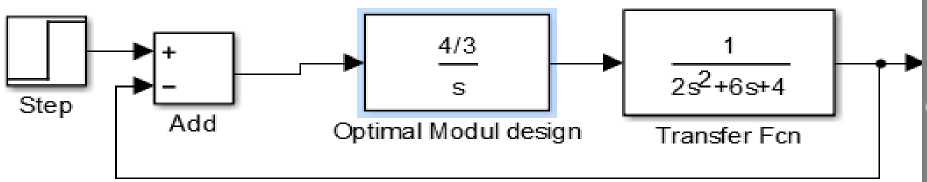
Optimal Module Criterion: Problem

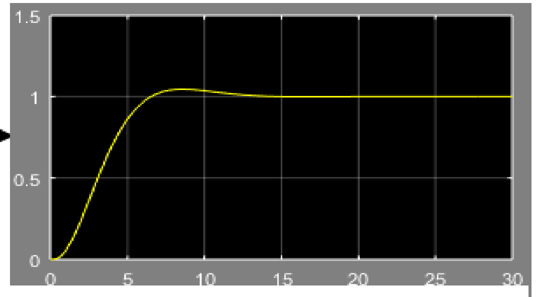
Determine I-controller using Optimal Module Criterion.

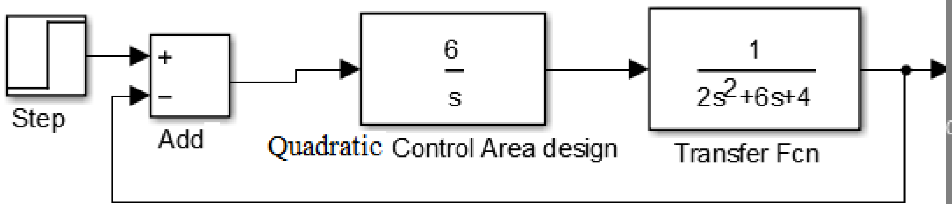
$$G_S(p) = \frac{1}{2p^2 + 6p + 4} \qquad G_R = \frac{r_{-1}}{p} \qquad z(t) = 1$$

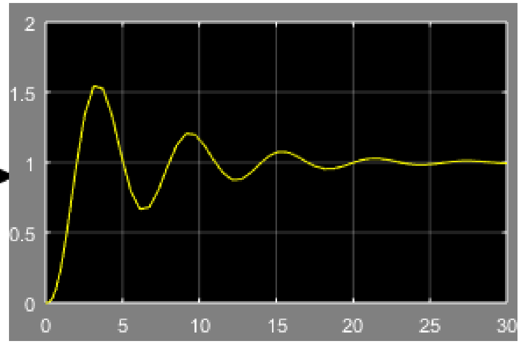
Solution: $r_{-1} = \dfrac{16}{12} = \dfrac{4}{3}$

Comparison: Opimum module and Quadratic control area









Design in state space: Problem

$$A = \begin{bmatrix} -0{,}05 & 0{,}05 & 0 \\ 0{,}05 & -0{,}1 & 0{,}05 \\ 0 & 0{,}05 & -0{,}1 \end{bmatrix} \quad B = \begin{bmatrix} 0{,}5 \\ 0 \\ 0 \end{bmatrix}$$

Determine stability of the system.

Determine matrix of controlability.

Determine matrix of observability.

Determine the feedback gain matrix K.

(Desired poles location is done:)

$$p_1 = -0{,}2 + 0{,}2j$$
$$p_2 = -0{,}2 - 0{,}2j$$
$$p_3 = -1$$

Luenberger observer: Illustrative example

Assume system:

$$A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\det(pI - A) = det \begin{bmatrix} p+3 & 1 \\ -2 & p \end{bmatrix} = p^2 + 3p + 2 = (p+1)(p+2) \qquad p_1 = -1 \qquad p_2 = -2$$

Eigenvalues of (*A*-*LC*) choice: -6; -6

Desired characteristic polynomial: $(p+6)^2 = p^2 + 12p + 36$

Characteristic polynomial of Luenberger observer:

$$\tilde{A} = A - LC = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} l_1 & 0 \\ l_2 & 0 \end{bmatrix} = \begin{bmatrix} -3 - l_1 & -1 \\ 2 - l_2 & 0 \end{bmatrix}$$

$$\det\left(pI - \tilde{A}\right) = det \begin{bmatrix} p + 3 + l_1 & 1 \\ -2 + l_2 & p \end{bmatrix} = p^2 + 3p + l_1 p + 2 - l_2 = p^2 + p(3 + l_1) + 2 - l_2$$

Comparing the both polynomials:

$$3 + l_1 = 12 \;\rightarrow\; l_1 = 9$$

$$L = \begin{bmatrix} 9 \\ -34 \end{bmatrix}$$

$$2 + l_2 = 36 \;\rightarrow\; l_2 = -34$$

Illustrative example in Matlab

You can use *place* for estimator gain*:* *L* = place(*A', C',P*)

The length of *P* must match the row size of *A*.

```
>> A=[-3 -1;2 0];
>> C=[1 0];
>> P=[-5.9 -6.1];               //The length of P must match the row size of A and be placed separately

>> L=place(A',C',P)'

L =

    9.0000
  -33.9900
```

References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.
ISBN: 81-7409-107-6

[2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing

[3] Matlab&Simulink: R2015a

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13. Discrete Time Control System

Table of Z-transform pairs

F(p)	f(t)	F(z)
$\frac{1}{p}$	$u_1(t)$	$\frac{z}{z-1}$
$\frac{1}{p^2}$	t	$\frac{Tz}{(z-1)^2}$
$\frac{1}{p^3}$	$\frac{1}{2}t^2$	$\frac{T^2 z(z+1)}{2(z-1)^3}$
$\frac{1}{p^{k+1}}$	$\frac{1}{k!}t^k$	$\lim_{a\to 0}\frac{(-1)^k}{k!}\frac{\partial k}{\partial a^k}\left(\frac{z}{z-e^{-aT}}\right)$
$\frac{1}{p+a}$	e^{-at}	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(p+a)^2}$	$t\cdot e^{-at}$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$\frac{a}{p(p+a)}$	$1-e^{-at}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\frac{a}{p^2(p+a)}$	$t-\frac{1-e^{-at}}{a}$	$\frac{Tz}{(z-1)^2}-\frac{z(1-e^{-aT})}{a(z-1)(z-e^{-aT})}$
$\frac{\omega_0}{p^2+\omega_0{}^2}$	$\sin\omega_0 t$	$\frac{z\sin\omega_0 T}{z^2-2z\cos\omega_0 T+1}$
$\frac{p}{p^2+\omega_0{}^2}$	$\cos\omega_0 t$	$\frac{z^2-z\cos\omega_0 T}{z^2-2z\cos\omega_0 T+1}$
$\frac{b-a}{(p+a)(p+b)}$	$e^{-at}-e^{-bt}$	$\frac{z}{z-e^{-aT}}-\frac{z}{z-e^{-bT}}$

F(p)	f(t)	F(z)
$\frac{(b-a)(p+c)}{(p+a)(p+b)}$	$(c-a)e^{-at}+(b-c)e^{-bt}$	$\frac{z(c-a)}{z-e^{-aT}}+\frac{z(b-c)}{z-e^{-bT}}$
$\frac{ab}{p(p+a)(p+b)}$	$1+\frac{b}{a-b}e^{-at}-\frac{a}{a-b}e^{-bt}$	$\frac{z}{z-1}+\frac{bz}{(a-b)(z-e^{-aT})}-\frac{az}{(a-b)(z-e^{-bT})}$
$\frac{a^2}{p(p+a)^2}$	$1-(1+at)e^{-at}$	$\frac{z}{z-1}-\frac{z}{z-e^{-aT}}-\frac{aTze^{-aT}}{(z-e^{-aT})^2}$
$\frac{a^2}{p^2(p+a)^2}$	$at-2+(at+2)e^{-at}$	$\frac{1}{a}\left[\frac{(aT+2)z-2z^2}{(z-1)^2}+\frac{2z}{z-e^{-aT}}+ +\frac{aTze^{-aT}}{(z-e^{-aT})^2}\right]$
$\frac{a^2b}{p(p+b)(p+a)^2}$	$\frac{2ab-b^2}{(a-b)^2}e^{-at}-\frac{a^2}{(a-b)^2}e^{-bt}+ +\frac{ab}{a-b}te^{-at}+1$	$\frac{z\left[2ab-b^2\right]}{(a-b)^2(z-e^{-aT})}-\frac{a^2z}{(a-b)^2(z-e^{-bT})}+ +\frac{abTze^{-aT}}{(a-b)(z-e^{-aT})^2}+\frac{z}{z-1}$
$\frac{a^2b^2}{p^2(p+a)(p+b)}$	$\frac{a^2}{a-b}e^{-bt}-(a+b)-\frac{b^2}{a-b}e^{-at}+abt$	$\frac{a^2z}{(a-b)(z-e^{-bT})}-\frac{z(a+b)}{z-1}- -\frac{b^2z}{(a-b)(z-e^{-aT})}+\frac{abTz}{(z-1)^2}$
$\frac{a^2b^2(p+c)}{p^2(p+a)(p+b)}$	$abct+\left[ab-c(a+b)\right]- -\frac{b^2(c-a)}{a-b}e^{-at}-\frac{a^2(b-c)}{a-b}e^{-bt}$	$\frac{abcTz}{(z-1)^2}+\frac{ab-c(a+b)z}{z-1}- -\frac{b^2z(c-a)}{(a-b)(z-e^{-aT})}-\frac{a^2z(b-c)}{(a-b)(z-e^{-bT})}$

Stability:Bilinear transformation $z = \frac{w+1}{w-1}$

Illustrative example: Determine the stability of the control system given by transfer function.

$$G(z) = \frac{z^2 + 0{,}2z - 0{,}5}{z^3 - 1{,}2z^2 + 0{,}45z - 0{,}05}$$

Characteristic equation:

$$A(z) = z^3 - 1{,}2z^2 + 0{,}45z - 0{,}05 = 0$$

Bilinear transformation:

$$\left(\frac{w+1}{w-1}\right)^3 - 1{,}2\left(\frac{w+1}{w-1}\right)^2 + 0{,}45\left(\frac{w+1}{w-1}\right) - 0{,}05 = 0$$

$$(w+1)^3 - 1{,}2(w+1)^2(w-1) + 0{,}45(w+1)(w-1)^2 - 0{,}05(w-1)^3 = 0$$

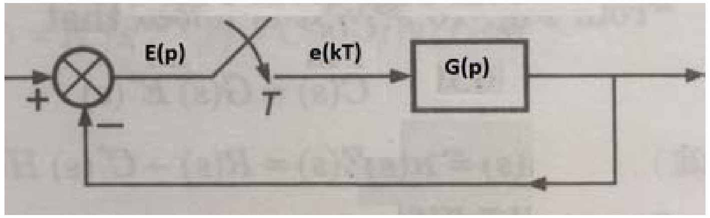
$$0{,}2w^3 + 1{,}5w^2 + 3{,}6w + 2{,}7 = 0$$

Hurwitz criterion:

$$H = \begin{vmatrix} 1{,}5 & 2{,}7 \\ 0{,}2 & 3{,}6 \end{vmatrix} = 4{,}86 > 0$$

Stability: Solved example

Determine the pulse transfer function and stability of the control system.



$$G_s(p) = \frac{5}{p(p+0,5}$$

$$G_s(p) = \frac{10}{p} + \frac{10}{p+0,5}$$

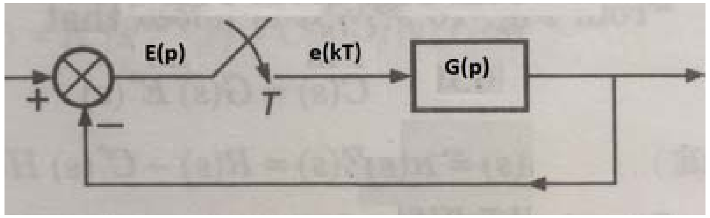
$$G_s(z) = 10\left[\frac{z}{z-1} - \frac{z}{(z-e^{-0,5T})}\right] = \frac{10z(1-e^{-0,5T})}{z^2 - z(1-e^{-0,5T}) + e^{-0,5T}}$$

$$G_w(z) = \frac{G_s(z)}{1+G_s(z)} = \frac{\dfrac{10z(1-e^{-0,5T})}{z^2 - z(1-e^{-0,5T}) + e^{-0,5T}}}{1+\dfrac{10z(1-e^{-0,5T})}{z^2 - z(1-e^{-0,5T}) + e^{-0,5T}}} = \frac{10z(1-e^{-0,5T})}{z^2 - z(11e^{-0,5T} - 9) + e^{-0,5T}}$$

$$A(z) = z^2 - z(11e^{-0,5T} - 9) + e^{-0,5T}$$

Stability: Solved example

Determine the pulse transfer function and stability of the control system.



$$G_s(p) = \frac{5}{p(p+0{,}5)}$$

$$A(z) = z^2 - z(11e^{-0{,}5T} - 9) + e^{-0{,}5T}$$

Fot the sampling time **T = 0,5**:

$$z^2 + 0{,}42z + 0{,}78 = 0$$

$z_{1,2} = -0{,}21 \pm 0{,}85i$ → System is stable, all the roots lie within unit circle of z-plane.

For the sampling time **T = 1,0**:

$$z^2 - \ 2{,}33z + 0{,}606 = 0$$

$z_1 = +2{,}10317$

$z_2 = +0{,}2983$ → System is unstable, one root is located outside of unit circle.

Conclusion: the sampling time can influence the stability of the system.

References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.
ISBN: 81-7409-107-6

[2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing

[3] Matlab&Simulink: R2015a

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14. Control Design in Discrete Time Domain

Example

Sampling time T=5s, consider zero order hold $H_0(p)$. Determine Z-transfer function of the system.

$$G_S(\mathrm{p}) = \frac{1}{(5\mathrm{p}+1)(\mathrm{p}+1)}$$

Solution:

$$G_c(\mathrm{z}) = Z\left\{\frac{1-\mathrm{e}^{-\mathrm{p}T}}{\mathrm{p}}\frac{1}{(5\mathrm{p}+1)(\mathrm{p}+1)}\right\} = \frac{0,5418z+0,0861}{z^2-0,3746z+0,0025}$$

Solution in Matlab:

1

```
>> num=1
num =
    1
>> den=[5 6 1]
den =
    5    6    1
```

2

```
>> [numd,dend]=c2dm(num,den,5,'zoh')
numd =
        0    0.5418    0.0860

dend =
    1.0000   -0.3746    0.0025
```

3

```
>> printsys(numd,dend,'z')
num/den =
        0.5418 z + 0.0860
     ------------------------
     z^2 - 0.3746 z + 0.0025
```

1.Solved example

Transfer function of the system: $G_S(p) = \frac{1}{(5p+1)(p+1)}$ Input: *w(t)* = 1

Zero-order hold: $H_0(p) = \frac{1}{p} - \frac{1}{p}e^{-pT}$

Sampling time: *T* = 5s

We need zero steady state deviation and final transient time.

Solution:

Total Z-transform of the continuous part of the control circuit: $G_c(z) = \frac{0,5418z + 0,0861}{z^2 - 0,3746z + 0,0025}$

Request conditions:

1. Zero steady state deviation: $1 - G_w(z) = (1 - z^{-1}) \cdot G(z) = (1 - z^{-1})(1 + g_1 z^{-1})$

$$g_0 = 1$$

2. Final transient time: $G_w(z) = P(z) \cdot R(z) = (0{,}5418z + 0{,}0861)\ r_2\ z^{-2}$

$$r_0 = 0$$
$$r_1 = 0$$

1. Solved example

Comparing coefficients of the both polynomials:

$$1 - (0{,}5418z + 0{,}0861)\, r_2\, z^{-2} = (1 - z^{-1})\,(1 + g_1 z^{-1})$$

$$0{,}5418 \cdot r_2 = 1 - g_1 \qquad r_2 = 1{,}5926$$

$$0{,}0861 \cdot r_2 = g_1 \qquad g_1 = 0{,}1371$$

Required control transfer function: $G_w(z) = 0{,}8621\, z^{-1} + 0{,}1371\, z^{-2}$

Discrete regulator: $D(z) = \dfrac{1}{G_c(z)} \cdot \dfrac{G_w(z)}{1 - G_w(z)} = \dfrac{1{,}5926 - 0{,}5965z^{-1} + 0{,}0039z^{-2}}{1 - 0{,}8629z^{-1} - 0{,}1371z^{-2}} = \dfrac{U(z)}{E(z)}$

Difference equation: $E(z) - 0{,}5965E(z)z^{-1} + 0{,}0025E(z)z^{-2} = U(z) - 0{,}8629U(z)z^{-1} - 0{,}1371U(z)z^{-2}$

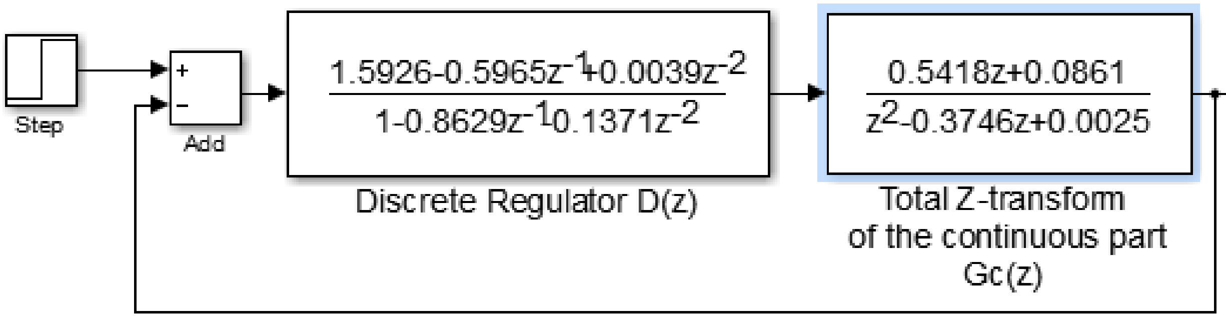
$$e(\mathrm{k}) - 0{,}5965e(k-1) + 0{,}0025e(k-2) = u(k) - 0{,}8629u(k-1) - 0{,}1371u(k-2)$$

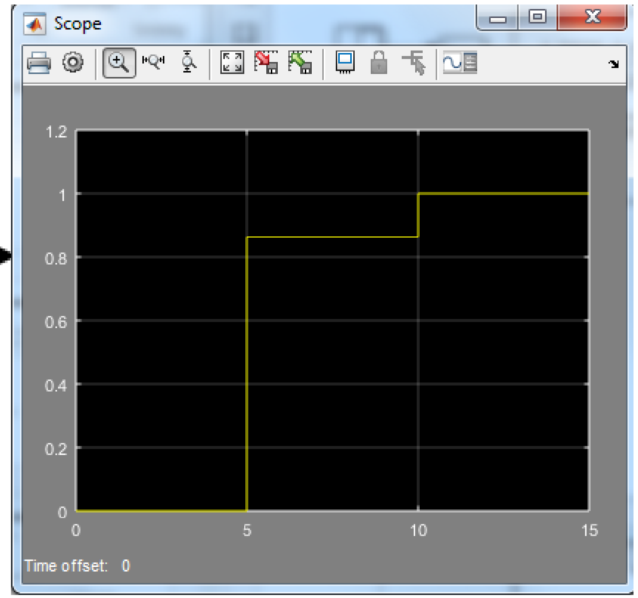
Control algorithm for actuating signal:

$$u(\mathrm{k}) = e(k) - 0{,}5965e(k-1) + 0{,}0025e(k-2) + 0{,}8629u(k-1) + 0{,}1371u(k-2)$$

1. Simulation in Matlab - Simulink

Block diagram and step response:





2. Solved example

Transfer function of the system: $G_s(p) = \frac{1}{(5p+1)(p+1)}$ Disturbance input: v(t) = 1

Zero-order hold: $H_0(p) = \frac{1}{p} - \frac{1}{p}e^{-pT}$

Sampling time: *T = 5s*

We need zero steady state deviation and final transient time.

Solution:

Total transfer function of the continuous part:

$$G_c(z) = Z\left\{\frac{1-e^{-pT}}{p}\frac{1}{(5p+1)(p+1)}\right\}_{T=5} = \frac{z-1}{z}\, Z\left\{\frac{1}{p\,(5p+1)(p+1)}\right\}_{T=5} = \frac{0,5418z+0,0861}{z^2-0,3746z+0,0025} = \frac{P(z)}{Q(z)}$$

Using Matlab:

```
[NUMd,DENd] = C2DM(NUM,DEN,Ts,'method')
[NUMd,DENd]=C2DM([1],[5 6 1],5,'zoh')
```

2. Solved example

Requirements:

1. The physical feasibility of the discrete regulator: $g_1 = p_1 = 0{,}5418$
2. Regulation to zero steady state deviation: $G_v = (1 - z^{-1})(g_0 + g_1 \cdot z^{-1} + \cdots) = (1 - z)\,(g_1 z^{-1} + g_2 z^{-2})$
3. The final duration of the transition process: $G_v = P(z)\cdot(r_0 + r_1 \cdot z^{-1} + \cdots\ ) = (\ 0{,}5418\ z^{-1} + 0{,}0861\ z^{-2}\ )\ (\ 1 + r_1 z^{-1}\ )$

Comparing coefficients of the both polynomials:

$$g_1 = 0{,}5418$$
$$g_2 = 0{,}0861$$
$$r_1 = -1$$

Required disturbance transfer function: $G_v = 0{,}5418 \cdot z^{-1} - 0{,}4557 \cdot z^{-2} - 0{,}0861 \cdot z^{-3}$

Discrete regulator:

$$D(z) = \frac{1}{G_V(z)} - \frac{1}{G_c(z)} = \frac{1}{P(z)R(z)} - \frac{Q(z)}{P(z)} = \frac{1 - Q(z)R(z)}{P(z)R(z)} =$$
$$= \frac{1{,}3746 - 0{,}3771 z^{-1} + 0{,}0025 z^{-2}}{0{,}5418 - 0{,}4537 z^{-1} - 0{,}0861 z^{-2}} = \frac{U(z)}{E(z)}$$

2. Solved example

Difference equation:

$$1,3746E(z)-0,3771E(z)z^{-1}+0,0025E(z)z^{-2}=0,5418U(z)-0,4537U(z)z^{-1}-0,0861U(z)z^{-2}$$

$$1,3746e(k)-0,3771e(k-1)+0,0025e(k-2)=0,5418u(k)-0,4537u(k-1)-0,0861u(k-2)$$

Control algorithm for actuating signal:

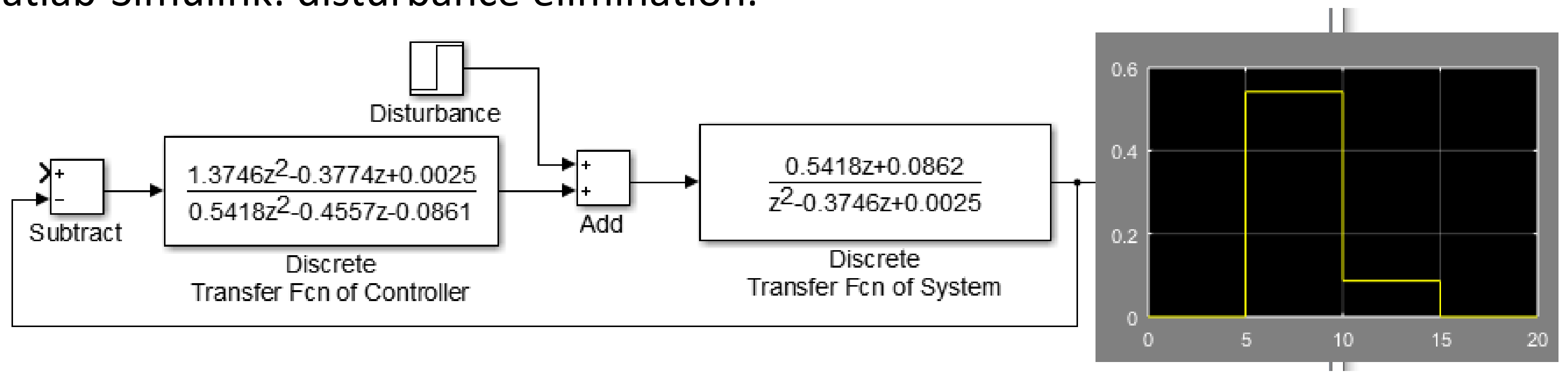
$$u(k)=\frac{1}{0,5418}\left(1,3746e(k)-0,3771e(k-1)+0,0025e(k-2)+0,4537u(k-1)+0,0861u(k-2)\right)$$

Control transfer function:

$$G_w(z)=\frac{D(z)G_c(z)}{1+D(z)G_c(z)}=G_V(z)D(z)=P(z)R(z)\frac{1-Q(z)R(z)}{P(z)R(z)}=$$

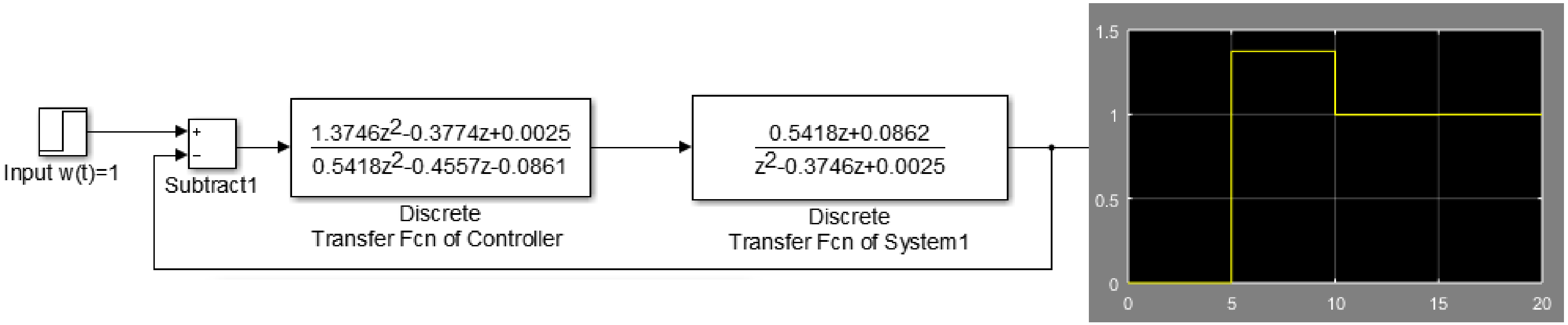
$$=1-Q(z)R(z)=1,3746-0,3771z^{-1}+0,0025z^{-2}$$

Simulation in Matlab-Simulink: disturbance elimination.



2. Simulation in Matlab-Simulilnk

Simulation in Matlab-Simulink: step response.



3. Solved example

For the system: $G_s = \dfrac{1}{(5p+1)(p+1)}$ was designed the continuous PI controller: $G_R = 0{,}5\dfrac{5p+1}{p}$

Determine corresponding PS – controller $D(z) = \dfrac{d_0 - d_1 z^{-1}}{1 - z^{-1}}$

Solution:

Rearrange into standard PI form: $G_R = 0,5\dfrac{5p+1}{p} = 2,5(1+\dfrac{1}{5p})$ $K = 2,5$ $T_i = 5$

Discrete controller: $d_0 = K(1-\dfrac{T}{T_i}) = 2,5(1-\dfrac{1}{5}) = 3$

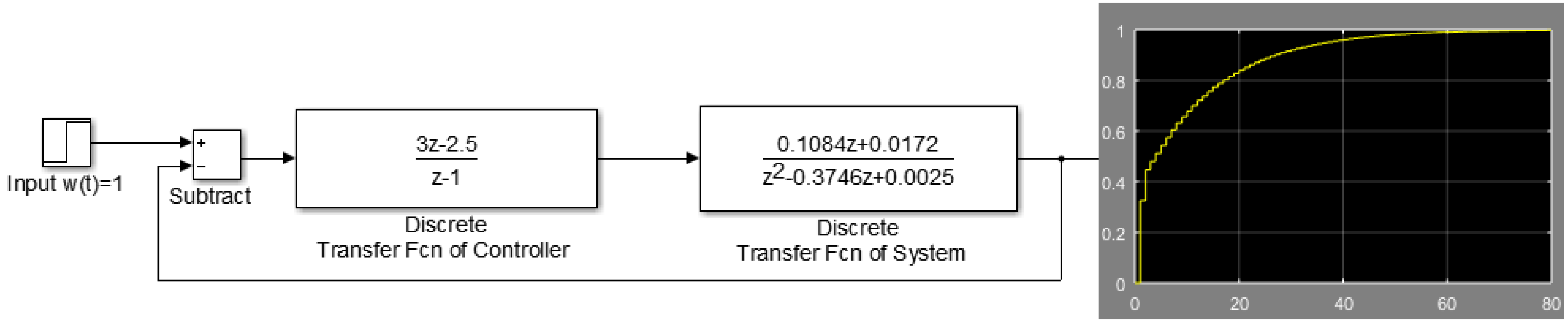
$d_1 = K = 2,5$

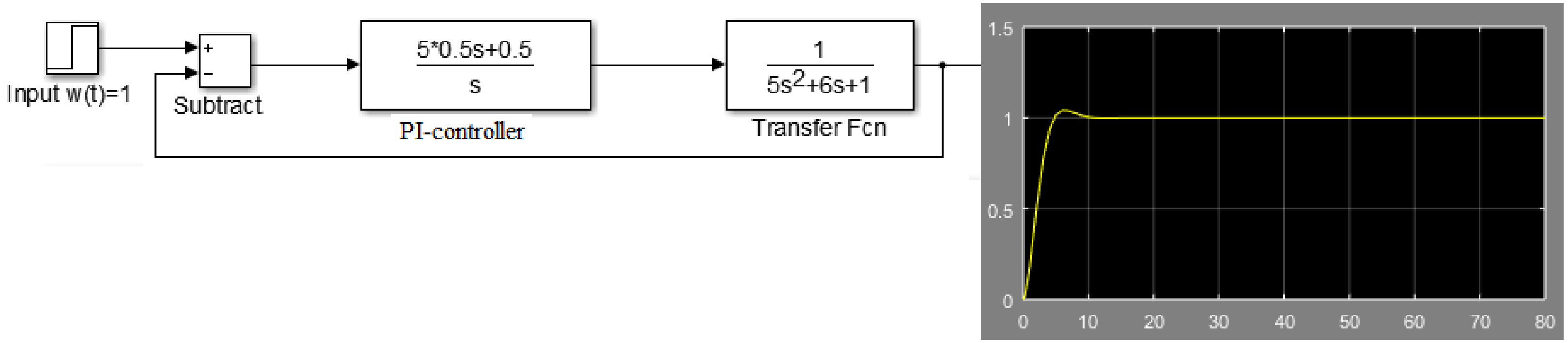
$D(z) = \dfrac{3-2,5z^{-1}}{1-z^{-1}} = \dfrac{U(z)}{E(z)}$

Actuating signal: $u(n) = u(n-1) + 3e(n) - 2,5e(n-1)$

3. Solved example

Simulation in Matlab-Simulink: step responses





References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.
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[2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing

[3] Matlab&Simulink: R2015a