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1.Laplace Transformation in Control System



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Use in Control System

- Describing of dynamic properties of a control system
- - controlled system and
- - actuators
- Solving integral-differential equation
- Creating transfer function of dynamic system

Definition formula of Laplace transform

Laplace transform belongs to integral transforms, which are defined by this common formula:

$$F(p) = \int_a^b K(p, t) \cdot f(t) dt$$

Definition formula of Laplace transform:

$$F(p) = L\{f(t)\} = \int_0^{\infty} f(t) \cdot e^{-pt} dt$$

- L is the symbol of Laplace transform
- $f(t)$ is an original function (standard time function)
- t is the real variable, usually time in our case
- $F(p)$ is a complex function of complex variable p
- p is transform variable, this is complex number $p \in \mathbb{C}$

Only one complex function $F(p)$ belongs to original function $f(t)$.

Only one original function $f(t)$ belongs to complex function $F(p)$.

Solved Examples

Find the Laplace transform of the following functions.

- Constant:** $f(t) = A, \quad t > 0$
 $L\{A\} = \frac{A}{p}$

$$1. L\{A\} = \int_0^{\infty} A \cdot e^{-pt} dt = A \left[-\frac{1}{p} \cdot e^{-pt} \right]_0^{\infty} = A \left(-\frac{1}{p} \cdot 0 + \frac{1}{p} \cdot 1 \right) = \frac{A}{p}$$
- Exponential function:** $f(t) = e^{-at}, \quad a > 0$
 $L\{e^{-at}\} = \frac{1}{p+a}$

$$2. L\{e^{-at}\} = \int_0^{\infty} e^{-at} \cdot e^{-pt} dt = \int_0^{\infty} e^{-(a+p)t} dt = \left[-\frac{1}{a+p} e^{-(a+p)t} \right]_0^{\infty} = 0 + \frac{1}{p+a}$$
- Sinusoidal functions:** $\sin \omega t = \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t})$ $\cos \omega t = \frac{1}{2} (e^{j\omega t} + e^{-j\omega t})$

$$3. L\{\sin \omega t\} = \frac{1}{2j} \int_0^{\infty} (e^{j\omega t} - e^{-j\omega t}) \cdot e^{-pt} dt = \frac{1}{2j} \left[\int_0^{\infty} e^{-(p-j\omega)t} dt - \int_0^{\infty} e^{-(p+j\omega)t} dt \right] =$$

$$\frac{1}{2j} \left(\frac{1}{p-j\omega} - \frac{1}{p+j\omega} \right) = \frac{\omega}{p^2 + \omega^2}$$

$$f(t) = \sin(\omega t) \Rightarrow F(p) = \frac{\omega}{p^2 + \omega^2}$$

$$f(t) = \cos(\omega t) \Rightarrow F(p) = \frac{p}{p^2 + \omega^2}$$

Basic Laplace Transform Theorems

- **1. Linearity** (transform of linear combination of functions)

$$L\left\{\sum c_i \cdot f_i(t)\right\} = \sum c_i \cdot L\{f_i(t)\} = \sum c_i \cdot F(p) \quad \text{where } c_i \text{ is constant}$$

Using theorem of linearity:

- L-transform of function multiplied by constant:

$$L\{A \cdot f(t)\} = A \cdot L\{f(t)\} = A \cdot F(p)$$

- L- transform of the sum of functions

$$L\{f_1(t) + f_2(t) + \cdots f_n(t)\} = L\{f_1(t)\} + L\{f_2(t)\} + \cdots L\{f_n(t)\} = F_1(p) + F_2(p) + \cdots F_n(p)$$

Basic Laplace Transform Theorems

- **2. Shifting** (multiplying of time function by an exponential)

$$L\{e^{at} \cdot f(t)\} = F(p - a)$$

- **3. Differentiation of L-transform** (multiplying of time function by the variable t)

$$L\{t \cdot f(t)\} = -\frac{d}{dp} F(p)$$

Examples:

$$L\{t\} = L\{t \cdot 1(t)\} = -\frac{d}{dp} L\{1(t)\} = -\frac{d}{dp} \frac{1}{p} = \frac{1}{p^2} \quad \text{ramp function}$$

$$L\{t^2\} = L\{t^2 \cdot 1(t)\} = \frac{2}{p^3} \quad \text{parabolic function}$$

Basic Laplace Transform Theorems

4. Shift in time (multiplying of transforms by an exponential)

$$L\{f(t-a)\} = L\{f(t-a) \cdot 1(t-a)\} = e^{-ap} \cdot L\{f(t)\} = e^{-ap} F(p)$$

Number $a \geq 0$ be a real constant

5. Differentiation of time function

a) first-order derivative $f^{(1)}(t)$: $L\left\{\frac{df}{dt}\right\} = pF(p) - f(0)$

We have to use integration by parts here: $L\left\{\frac{df}{dt}\right\} = \int_0^{\infty} \frac{df}{dt} \cdot e^{-pt} dt = \left[f(t) \cdot e^{-pt}\right]_0^{\infty} + p \cdot \int_0^{\infty} f(t) \cdot e^{-pt} dt = -f(0) + pF(p)$

b) second-order derivative $f^{(2)}(t)$: $L\left\{\frac{d^2 f}{dt^2}\right\} = p^2 F(p) - p \cdot f(0) - f^{(1)}(0)$

It can be generalized:

$$L\left\{\frac{d^{(n)} f}{dt^{(n)}}\right\} = p^n \cdot F(p) - p^{n-1} f(0) - p^{n-2} \cdot f^{(1)}(0) - \dots - p \cdot f^{(n-2)}(0) - f^{(n-1)}(0)$$

Basic Laplace Transform Theorems

6. Integration of time function

$$L\left\{\int_0^t f(t)dt\right\} = \frac{1}{p} F(p)$$

We also need the integration by parts here:

$$\int_0^{\infty} \left[\int_0^t f(t)dt \right] \cdot e^{-pt} dt = \left[-\frac{1}{p} \cdot e^{-pt} \int_0^t f(t)dt \right]_0^{\infty} + \frac{1}{p} \cdot \int_0^{\infty} f(t) \cdot e^{-pt} dt = \frac{1}{p} \cdot F(p)$$

Laplace transform technique transforms a time domain differential equation into a frequency domain algebraic equation.

The final-value and initial-value theorems.

- Two theorems, that can be used to find the value of the time-domain function at two extremes, $t = 0$ and $t = \infty$ without having to do inverse L-transform.

1. The initial theorem: $f(t \rightarrow 0) = ?$
$$\lim_{t \rightarrow 0} f(t) = \lim_{p \rightarrow \infty} p \cdot F(p)$$

2. The final value theorem: $f(t \rightarrow \infty) = ?$
$$\lim_{t \rightarrow \infty} f(t) = \lim_{p \rightarrow 0} p \cdot F(p)$$

The final value theorem gives the final value ($t \rightarrow \infty$) of a time function its Laplace transform and as such very useful in the analysis of control systems. However, if the denominator of $p \cdot F(p)$ has any root having real part as zero or positive, then the final value theorem is not valid.

Example

1. Consider $f(t) = e^t$. Clearly we have $f(\infty) = \infty$.

$$F(p) = 1/(p-1) \rightarrow \lim_{p \rightarrow 0} p \cdot F(p) = \lim_{p \rightarrow 0} p \cdot \frac{1}{p-1} = 0 \quad \text{which is not equal to } f(\infty).$$

2. Consider $f(t) = \sin 2t$. Value $f(t \rightarrow \infty)$ is not defined.

$$F(p) = \frac{2}{p^2 + 4} \rightarrow \lim_{p \rightarrow 0} p \cdot F(p) = \lim_{p \rightarrow 0} p \cdot \frac{2}{p^2 + 4} = 0 \quad \text{The formula is not valid again.}$$

Conclusion:

If all poles of $p \cdot F(p)$ have negative real parts, then the final-value theorem is valid.

Final value theorem - examples

- Find $f(\infty)$ if $F(p)$ is given by:

$$F(p) = \frac{1}{p-5}$$

$$F(p) = \frac{p-2}{p(p+1)}$$

$$F(p) = \frac{p+5}{p(p+2-j)(p+2+j)}$$

$$F(p) = \frac{1}{(p+j)(p-j)}$$

$$F(p) = \frac{p+10}{(p+2)(p+3)(p+4)} \quad f(\infty) = 0$$

$$F(p) = \frac{p+1}{p(p^2-1)} \quad f(\infty) = 1$$

$$F(p) = \frac{1}{p^3 + 2p^2 + 9p + 68}$$

$$F(p) = \frac{p^3 + p^2 - 2}{p^4 + 2p^3 + 2p^2 - 2p - 3}$$

Laplace Transform Common to control problems

1. Step function:

$$f(t) = A \cdot u(t) \Rightarrow F(p) = \frac{A}{p} \text{ where}$$

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \Rightarrow L\{u(t)\} = \frac{1}{p}$$

Definition of unit step function is also called **Heaviside function** in mathematics.

2. Ramp function:

$$f(t) = a \cdot t, \quad \text{for } t \geq 0 \text{ and } a = \text{const.}$$

$$L\{a \cdot t\} = a \cdot L\{f(t)\} = a \frac{1}{p^2}$$

Laplace Transform Common to control problems

3. Time delay function

It is called also dead time function or transport-lag function.

It is defined such that an original function $f(t)$ is shifted in time t_0 .

This time delay function can be written as:

$$f(t-t_0) = \begin{cases} 0 & \text{for } t-t_0 < 0 \\ f(t-t_0) & \text{for } t-t_0 > 0 \end{cases}$$

$$L \{ f(t-t_0) \} = L \{ f(t-t_0) \cdot 1(t-t_0) \} = e^{-t_0 p} \cdot L \{ f(t) \} = e^{-t_0 p} F(p)$$

Laplace Transform Common to control problems

4. a) Rectangular-pulse function

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ A & \text{for } 0 < t < T \\ 0 & \text{for } t > T \end{cases} = A [u(t) - u(t - T)] \quad L\{f(t)\} = \frac{A}{p} (1 - e^{-pT})$$

b) Unit rectangular –pulse function

We can define a rectangular pulse such that the area is a unity.

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1/T & \text{for } 0 < t < T \\ 0 & \text{for } t > T \end{cases} = \frac{1}{T} [u(t) - u(t - T)]$$
$$L\left\{\frac{1}{T} [u(t) - u(t - T)]\right\} = \frac{1}{T} \left(\frac{1}{p} - \frac{1}{p} e^{-pT}\right) = \frac{1}{pT} (1 - e^{-pT})$$

Laplace Transform Common to control problems

5. Impulse function

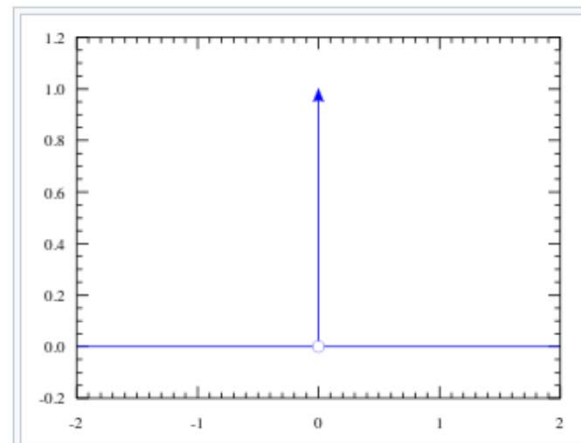
The impulse function is called Dirac function (or simply delta function). We can consider the impulse function as the units rectangular function as T shrinks to zero while height $1/T$ goes to infinity.

$$f(t) = \delta(t)$$

$$L \{ \delta(t) \} = 1$$

$$L \{ A \cdot \delta(t) \} = A$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$



Schematic representation of the Dirac delta function by a line surmounted by an arrow. The height of the arrow is usually meant to specify the value of any multiplicative constant, which will give the area under the function. The other convention is to write the area next to the arrowhead.

https://en.wikipedia.org/wiki/Dirac_delta_function

Table of Laplace transform pairs

$F(p) = \mathcal{L}\{f(t)\}$	$f(t)$	$F(p) = \mathcal{L}\{f(t)\}$	$f(t)$
1	$\delta(t)$	$\frac{a}{p^2 - a^2}$	$\sinh at$
$\frac{1}{p}$	1	$\frac{p}{p^2 - a^2}$	$\cosh at$
$\frac{1}{p^n}$	$\frac{t^{n-1}}{(n-1)!}$	$\frac{2\omega p}{(p^2 + \omega^2)^2}$	$t \sin \omega t$
$\frac{1}{p - a}$	e^{at}	$\frac{p^2 - \omega^2}{(p^2 + \omega^2)^2}$	$t \cos \omega t$
$\frac{1}{(p - a)^n}$	$\frac{t^{n-1} e^{at}}{(n-1)!}$	$\frac{\omega}{(p - a)^2 + \omega^2}$	$e^{at} \sin \omega t$
$\frac{\omega}{p^2 + \omega^2}$	$\sin \omega t$	$\frac{p - a}{(p - a)^2 + \omega^2}$	$e^{at} \cos \omega t$
$\frac{p}{p^2 + \omega^2}$	$\cos \omega t$		

<http://cam.zcu.cz/~danek/Students/ME4/slovniky/LT.pdf>

Solved examples

1. $f(t) = 2 + 3te^{-2t} - 4t^2e^{-3t}$

$$F(p) = \frac{2}{p} + \frac{3}{(p+2)^2} - \frac{4 \cdot 2}{(p+3)^3}$$

$$e^{at}f(t) \triangleq F(p-a), \quad a = -2, \quad a = -3$$

$$1 \triangleq \frac{1}{p}, \quad t \triangleq \frac{1}{p^2}, \quad t^2 \triangleq \frac{2}{p^3}.$$

2. $f(t) = 3 \sin 2t - 5 \cos 2t$

$$F(p) = \frac{3 \cdot 2}{p^2+4} - \frac{5p}{p^2+4} = \frac{6-5p}{p^2+4}$$

$$\sin 2t \triangleq \frac{2}{p^2+4}, \quad \cos 2t \triangleq \frac{p}{p^2+4}.$$

3. $f(t) = 3t - \sin 2t$

$$F(p) = \frac{3}{p^2} - \frac{2}{p^2+4}$$

$$t \triangleq \frac{1}{p^2}, \quad \sin 2t \triangleq \frac{2}{p^2+4}.$$

4. $f(t) = t^2 - 1 + 3e^{-t} + \cos 2t$

$$F(p) = \frac{2}{p^3} - \frac{1}{p} + \frac{3}{p+1} + \frac{p}{p^2+4}$$

$$t^n \triangleq \frac{n!}{p^{n+1}}, \quad e^{-2t} \triangleq \frac{1}{p+2}, \quad \cos 2t \triangleq \frac{p}{p^2+4}.$$

5. $f(t) = 4e^t + 2e^{-3t} + \sin 2t$

$$F(p) = \frac{4}{p-1} + \frac{2}{p+3} + \frac{2}{p^2+4}$$

$$e^{at}f(t) \triangleq F(p-a), \quad a = 1, \quad a = -3 \quad 1 \triangleq \frac{1}{p}, \quad \sin 2t \triangleq \frac{2}{p^2+4}.$$

6. $f(t) = (2t+5)e^{-2t} + 3 \cos t - 2 \sin 3t$

$$F(p) = \frac{2}{(p+2)^2} + \frac{5}{p+2} + \frac{3p}{p^2+1} - \frac{6}{p^2+9}$$

$$e^{-2t}f(t) \triangleq F(p+2)$$

$$1 \triangleq \frac{1}{p}, \quad t \triangleq \frac{1}{p^2}, \quad \cos t \triangleq \frac{p}{p^2+1}, \quad \sin 3t \triangleq \frac{3}{p^2+9}.$$

7. $f(t) = t(\sin 2t + 4 \cos 2t)$

$$F(p) = -\left(\frac{2}{p^2+4} + \frac{4p}{p^2+4}\right)' = \frac{4p^2+4p-16}{(p^2+4)^2}$$

$$tf(t) \triangleq -F'(p) \quad \sin 2t \triangleq \frac{2}{p^2+4}, \quad \cos 2t \triangleq \frac{p}{p^2+4}.$$

8. $f(t) = (t+2) \cos 3t$

$$F(p) = -\left(\frac{p}{p^2+9}\right)' + \frac{2p}{p^2+9} = \frac{p^2-9}{(p^2+9)^2} + \frac{2p}{p^2+9} = \frac{2p^3+p^2+18p-9}{(p^2+9)^2}$$

$$tf(t) \triangleq -F'(p) \quad \cos 3t \triangleq \frac{p}{p^2+9}.$$

9. $f(t) = (3t^2 + 2t - 1)e^{-t} + (t+1) \sin 2t$

$$F(p) = \frac{6}{(p+1)^3} + \frac{2}{(p+1)^2} - \frac{1}{p+1} - \left(\frac{2}{p^2+4}\right)' + \frac{2}{p^2+4} = \frac{6}{(p+1)^3} + \frac{2}{(p+1)^2} - \frac{1}{p+1} + \frac{4p}{(p^2+4)^2} + \frac{2}{p^2+4}$$

$$e^{-t}f(t) \triangleq F(p+1), \quad tf(t) \triangleq -F'(p)$$

$$1 \triangleq \frac{1}{p}, \quad t \triangleq \frac{1}{p^2}, \quad t^2 \triangleq \frac{2}{p^3}, \quad \sin 2t \triangleq \frac{2}{p^2+4}.$$

10. $f(t) = te^{-3t} + (t-5) \cos 3t$

$$F(p) = \frac{1}{(p+3)^2} - \left(\frac{p}{p^2+9}\right)' - \frac{5p}{p^2+9} = \frac{1}{(p+3)^2} + \frac{p^2-9}{(p^2+9)^2} - \frac{5p}{p^2+9}$$

$$e^{-3t}f(t) \triangleq F(p+3), \quad tf(t) \triangleq -F'(p) \quad \text{a vzorce } t \triangleq \frac{1}{p^2}, \quad \cos 3t \triangleq \frac{p}{p^2+9}.$$

Revision

1. Write the definition formula of L-transform.
2. Do you remember Laplace transform of this following functions ?

$$L\{1(t)\} =$$

$$L\{e^{-at}\} =$$

$$L\{\delta(t)\} =$$

$$L\{f'(t)\} =$$

$$L\left\{\int_0^{\infty} f(t) dt\right\} =$$

Revision

3. Write L-transform of

- the unit step
- a constant function
- the ramp function
- an exponential function
- Dirac function
- derivation: $L\{x'(t)\} =$
- function shifted in time: $L\{f(t-a)\} =$

References

- [1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.
ISBN: 81-7409-107-6
- [2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing
- [3] Matlab&Simulink: R2015a

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2. Inverse Laplace Transform

Inverse Laplace transform of rational function

Find the inverse Laplace transform of the following function using partial fraction expansion:

1. Function has distinct real roots:

$$F(p) = \frac{6p^2 - 12}{(p+1)(p+2)(p-2)} = \frac{A}{p+1} + \frac{B}{p+2} + \frac{C}{p-2}$$

$$A = \lim_{p \rightarrow -1} (p+1) \cdot F(p) = \lim_{p \rightarrow -1} \frac{6p^2 - 12}{(p+2)(p-2)} = \frac{-6}{-3} = 2$$

$$B = \lim_{p \rightarrow -2} (p+2) \cdot F(p) = \lim_{p \rightarrow -2} \frac{6p^2 - 12}{(p+1)(p-2)} = 3$$

$$C = \lim_{p \rightarrow 2} (p-2) \cdot F(p) = \lim_{p \rightarrow 2} \frac{6p^2 - 12}{(p+2)(p+1)} = 1$$

$$F(p) = \frac{6p^2 - 12}{(p+1)(p+2)(p-2)} = \frac{2}{p+1} + \frac{3}{p+2} + \frac{1}{p-2}$$

Then the inverse L-transform of function is the sum of inverse L-transform of particular fractions.

$$f(t) = 2 \cdot e^{-t} + 3 \cdot e^{-2t} + e^{+2t}$$

(this is the sum of three exponential functions)

Inverse Laplace transform of rational function

Find the inverse Laplace transform of the **following** function using partial fraction expansion:

2. Function has two complex conjugated roots:

$$F(p) = \frac{p+5}{p^2+4p+13}$$

Roots can be found using Matlab command: `>> p = roots(num, den)`

$$p_{1,2} = -2 \pm 3j$$

$$F(p) = \frac{A}{(p+2-3j)} + \frac{B}{(p+2+3j)}$$

$$A = \lim_{p \rightarrow -2+3j} (p+2-3j) \frac{p+5}{(p+2-3j)(p+2+3j)} = \frac{-2+3j+5}{-2+3j+2+3j} = \frac{3+3j}{6j} = \frac{3-3j}{6} = \frac{1}{2}(1-j)$$

$$B = \frac{1}{2}(1+j)$$

Computation of coefficient B is not necessary, because it must equal to the complex-conjugate of A.

Then:

$$F(p) = \frac{\frac{1}{2}(1-j)}{(p+2-3j)} + \frac{\frac{1}{2}(1+j)}{(p+2+3j)}$$

Then for $t > 0$:

$$f(t) = \frac{1}{2}(1-j) \cdot e^{(-2+3j)t} + \frac{1}{2}(1+j) \cdot e^{(-2-3j)t} = \frac{1}{2} \cdot e^{-2t} \cdot [2 \cdot (\cos 3t + \sin 3t)]$$

You can see that the function $f(t)$ is the sum of two harmonic functions.

$$\begin{aligned} f(t) &= \frac{1}{2}(1-j) \cdot e^{(-2+3j)t} + \frac{1}{2}(1+j) \cdot e^{(-2-3j)t} = \frac{1}{2} \cdot e^{-2t} [(1-j) \cdot e^{3jt} + (1+j) \cdot e^{-3jt}] = \\ &= \frac{1}{2} e^{-2t} (-je^{3jt} + je^{-3jt} + e^{3jt} + e^{-3jt}) = \frac{1}{2} e^{-2t} \left[2 \frac{1}{2j} (e^{j3t} - e^{-j3t}) + 2 \frac{1}{2} (e^{j3t} + e^{-j3t}) \right] = \end{aligned}$$

$$= \frac{1}{2} e^{-2t} (2 \sin 3t + 2 \cos 3t)$$

Inverse Laplace transform of rational function.

Find the inverse Laplace transform of the following function using partial fraction expansion:

3. Function has repeated roots:

$$F(p) = \frac{2}{(p+1)^3(p+2)} = \frac{A_1}{(p+1)} + \frac{A_2}{(p+1)^2} + \frac{A_3}{(p+1)^3} + \frac{B}{p+2}$$

We can state the general formula for computing A_i .

$$A_i = \frac{1}{(m-i)!} \lim_{p \rightarrow p_i} \frac{d^{m-1}}{dp^{m-1}} [(p-p_i)^m \cdot F(p)]$$

where $m = 3$ (which is the number of repeated roots)

$$A_1 = \frac{1}{(3-1)!} \lim_{p \rightarrow -1} \frac{d^2}{dp^2} \left[(p+1)^3 F(p) \right] = \frac{1}{2} \lim_{p \rightarrow -1} \frac{d^2}{dp^2} \frac{2}{(p+2)} = \frac{1}{2} \lim_{p \rightarrow -1} \frac{+4}{(p+2)^3} = +2$$

$$A_2 = \frac{1}{(3-2)!} \lim_{p \rightarrow -1} \frac{d}{dp} \left[(p+1)^3 F(p) \right] = \lim_{p \rightarrow -1} \frac{d}{dp} \frac{2}{(p+2)} = \lim_{p \rightarrow -1} \frac{-2}{(p+2)^2} = -2$$

$$A_3 = \lim_{p \rightarrow -1} (p+1)^3 F(p) = 2$$

$$B = \lim_{p \rightarrow -2} (p+2)F(p) = -2$$

$$\text{Then: } F(p) = \frac{2}{(p+1)} + \frac{-2}{(p+1)^2} + \frac{2}{(p+1)^3} + \frac{-2}{p+2} \quad \Rightarrow$$

$$f(t) = 2 \cdot e^{-t} - 2t \cdot e^{-t} + 2 \frac{t^2}{2} \cdot e^{-t} - 2 \cdot e^{-2t} = 2 \left[\left(1 - t + \frac{t^2}{2} \right) \cdot e^{-t} - e^{-2t} \right]$$

Problem solving:

Find the inverse Laplace transform of the following functions:

$$\text{a) } F(p) = \frac{6p}{(p+1)(p+2)(p-2)}$$

$$\text{b) } F(p) = \frac{6}{(p+1)(p+2)(p+3)}$$

Answer :

$$\text{a) } 2.e^{-t} - 3.e^{-2t} + e^{+2t}$$

$$\text{b) } 3.e^{-t} - 6.e^{-2t} + 3.e^{-3t}$$

Problem solving:

Find the inverse Laplace transform of the following functions:

$$\text{a) } F(p) = \frac{1}{p(p+1)}$$

$$\text{b) } F(p) = \frac{1}{p^2 + 4p + 8}$$

$$\text{c) } F(p) = \frac{5}{p \cdot (p^2 + 4p + 5)}$$

$$\text{d) } F(p) = \frac{p+6}{p \cdot (p^2 + 4p + 3)}$$

Solution:

$$\text{a) } f(t) = 1 - e^{-t}$$

$$\text{b) } f(t) = \frac{1}{2}e^{-2t}\sin(2t)$$

$$\text{c) } f(t) = 1 - e^{-2t}\cos t - 2e^{-2t}\sin t$$

$$\text{d) } f(t) = 2 - 2,5e^{-t} + \frac{1}{2}e^{-3t}$$

Differential equation solving using Laplace Transform

Find the L-transform of the differential equation given below and hence evaluate the time solution of the it, when the initial conditions are $y(0) = 0$, $y'(0) = 6$.

$$y''(t) + 5y'(t) + 6y(t) = 12e^t$$

Solution in L-transform:

$$p^2 \cdot Y(p) - p \cdot y(0) - y(0) + 5 \cdot (p \cdot Y(p) - y(0)) + 6 \cdot Y(p) = \frac{12}{p-1}$$

$$p^2 \cdot Y(p) - 6 + 5 \cdot p \cdot Y(p) + 6 \cdot Y(p) = \frac{12}{p-1}$$

$$Y(p) \cdot (p^2 + 5p + 6) = \frac{6p + 6}{p-1}$$

$$Y(p) = \frac{6p + 6}{(p-1)(p^2 + 5p + 6)}$$

Solution in time domain:

$$y(t) = L^{-1}\{Y(p)\} = L^{-1}\left\{\frac{6 \cdot p + 6}{(p-1)(p^2 + 5p + 6)}\right\} = L^{-1}\left\{\frac{1}{p-1} + \frac{2}{p+2} - \frac{3}{p+3}\right\}$$

$$y(t) = e^{+t} + 2e^{-2t} - 3e^{-3t}$$

Obtain the solution of differential equation given below:

$$\frac{d^2x}{dt^2} + 2 \frac{dx}{dt} + 2x = 0 \quad \text{given } x(0)=0 \text{ and } x'(0)=1$$

$$L \left\{ \frac{d^2x}{dt^2} \right\} + L \left\{ 2 \frac{dx}{dt} \right\} + L[2x] = 0$$

$$[p^2X(p) - p \cdot x(0) - x(0)] + [2pX(p) - x(0)] + 2X(p) = 0$$

$$X(p) = \frac{1}{p^2 + 2p + 2} = \frac{1}{(p + 1)^2 + 1^2}$$

$$L^{-1}\{X(p)\} = L^{-1} \left\{ \frac{1}{p^2 + 2p + 2} \right\} = L^{-1} \left\{ \frac{1}{(p + 1)^2 + 1^2} \right\}$$

$$x(t) = e^{-1} \cdot \sin t$$

Obtain the solution of differential equation given below:

$$2 \frac{dx}{dt} + 8x = 10; \text{ given } x(0) = 2$$

Taking L-transform of both sides the following equation is obtained:

$$L \left\{ 2 \frac{dx}{dt} + 8x \right\} = L\{10\}$$

$$2[pX(p) - x(0)] + 8X(p) = \frac{10}{p}$$

Substituting $x(0) = 2$

$$2[pX(p) - 2] + 8X(p) = \frac{10}{p}$$

Simplifying:

$$X(p) = \frac{2p + 5}{p(p + 4)}$$

Using partial fraction expansion:

$$X(p) = \frac{k_1}{p} + \frac{k_2}{p + 4} = \frac{1,25}{p} + \frac{0,75}{p + 4}$$

Taking inverse L-transform of both sides:

$$x(t) = 1,25 + 0,75e^{-4t}$$

Introduction to Symbolic Math Toolbox

Creating symbolic variables

- constructing real symbolic variables
`>> t=sym('t','real')`
- constructing positive symbolic variables
`>> a=sym('a','positive')`
- using the shortcut for creating symbolic variables
`>> syms p p1 p2`
- creating real and positive symbolic variables
`>> syms t real`
`>> syms m n positive`
`>> p=sym('p')`
- list current variables
`>> who`

Special function: Heaviside, Dirac, exponential

```
>> h=sym('heaviside(t)')
```

```
h =
```

```
heaviside(t)
```

```
>> d=sym('dirac(t)')
```

```
d =
```

```
dirac(t)
```

```
>> e=sym('exp(-5*t)')
```

```
e =
```

```
exp(-5*t)
```

Laplace Transform in Matlab

Transform of the unit step

```
>> Syms p t  
>> H=laplace(h,t,p)  
H =  
1/p
```

Transform of the unit impulse

```
>> D=laplace(d,t,p)  
D =  
1
```

Transform of the exponential function

```
>> syms a  
>> F=laplace('exp(a*t)',t,p)  
F =  
1/(p - a)
```

Inverse Laplace Transform in Matlab

```
>> hi=ilaplace(H,p,t)
```

```
hi =
```

```
1
```

```
>> di=ilaplace(D,p,t)
```

```
di =
```

```
dirac(t)
```

```
>> fi=ilaplace('1/(p+2)',p,t)
```

```
fi =
```

```
exp(-2*t)
```

```
>> f = ilaplace(F,p,t)
```

```
f =
```

```
exp(a*t)
```

Problem solving

Find the inverse Laplace transform of the following functions using Matlab.

- $f_1(t) = e^{-8t}$
- $f_2(t) = \sin(-8t)$
- $f_3(t) = \cos(-8t)$
- $f_4(t) = 10$
- $f_5(t) = t$
- $f_6(t) = t^2$

Laplace transform of rectangular pulse in Matlab

```
>> syms a positive;
```

```
>> syms T real;
```

```
>> h=sym('heaviside(t)');
```

```
>> h5= sym('heaviside(t-T)');
```

```
>> HH=laplace(h-h5,t,p);
```

or

```
>> hh=h-subs(h,t,t-T);
```

```
>> HH=laplace(A*hh,t,p);
```

Limits theorems of L-transform in Matlab

The initial value theorem of $f(t) = e^{-3t} \Rightarrow F(p) = \frac{1}{p+3}$

```
>> syms t,p;  
>> f=sym('exp(-3*t)');  
F=sym('1/(p+3)');
```

- In time domain:

```
>> f0=limit(f,t,0)  
f0 =  
1
```

- In Laplace transform:

```
>> F0=limit(F*p,p,inf)  
F0 =  
1
```

The final value theorem in Matlab:

- In time domain:

```
>> finf=limit(f,t,inf)
```

```
finf =  
0
```

- In Laplace transform

```
>> Finf=limit(F*p,p,0)
```

```
Finf =  
0
```

References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.

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[2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing

[3] Matlab&Simulink: R2015a

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3. Introduction to Matlab

Introduction to Matlab

- Matlab offers tools for technical computing and model-based design.
- Widely used throughout industry, government and academia.
- These products are accelerating the pace of discovery, innovation, development and learning in engineering and science.

Tools useful for control system design and analysis

MATLAB – the language of technical computing

- Control System Toolbox
- System Identification Toolbox
- Fuzzy logic toolbox
- Robust Control Toolbox
- Model predictive Control Toolbox
- Simulink Control Design
- Symbolic Math Toolbox

SIMULINK – simulation and model based design

The MATLAB product family

- Provides a high-level programming language, an interactive technical computing environment, and function for algorithm development, data analysis, data visualisation and numeric computation.
- MATLAB products enable a wide range of computationally intensive tasks, including filter design, statistics, flight test analysis and spectral analysis.

The SIMULINK product family

- It is an intensive block-diagram environment for simulation and Model-Based Design. Its graphical tools enable engineers to accurately describe, explore and implement the behavior of control, signal processing, image processing, communications, and physical systems.
- Simulink and related products support key elements of the development process for embedded systems, including requirements capture and specification, design, implementation and test and verification.

Matrixes and vectors in Matlab

- All variables are considered as matrix.
- Create a matrix or a vector: `>> A=[1 2 ; 3 4]`

A =

1 2

3 4

- A column vector is represented as an $m \times 1$ matrix. A row vector is represented as a $1 \times n$ matrix.
- A is the name of variable - there is a distinction between capital letter and small letter.
- The components are in the square brackets.
- The components are separated by blank space and the rows are separated by semicolon or back space.
- Matrix and vector components must be [arithmetical expressions](#) (numbers and/or symbolic expressions).

Special types of matrixes

- Identity matrix: `eye(n)` generates a rectangular matrix $n \times n$ with ones on the main diagonal

```
>> I=eye(2)
```

```
I =
```

```
1  0
```

```
0  1
```

- Zero matrix: `zero(m,n)` generates a rectangular matrix $m \times n$ of zeros

```
>> Z=zeros(2,3)
```

```
Z =
```

```
0  0  0
```

```
0  0  0
```

Special types of matrixes

- Ones matrix: `ones(n)` generates a rectangular matrix $n \times n$ of ones.

```
>> O=ones(2)
```

```
O =
```

```
 1  1
```

```
 1  1
```

- Random matrix: `rand(m,n)` returns an $n \times m$ matrix of random numbers.

```
>> R=rand(2,3)
```

```
R =
```

```
0.8147  0.1270  0.6324
```

```
0.9058  0.9134  0.0975
```

Matrix operations

We can do matrix arithmetic using the standard arithmetical operators.

Addition

```
>> A=[1 2;3 4]
```

```
A =
```

```
1 2
```

```
3 4
```

```
>> B=[2 2; 3 3]
```

```
B =
```

```
2 2
```

```
3 3
```

```
>> C=A+B
```

```
C =
```

```
3 4
```

```
6 7
```

Substraction

```
>> C1=A-B
```

```
C1 =
```

```
1 0
```

```
0 1
```

Matrix operations

We can do matrix arithmetic using the standard arithmetical operators.

Multiplicative operations - based on definition

```
>> A=[1 2;3 4]
```

```
A =
```

```
1 2
```

```
3 4
```

```
>> B=[2 2; 3 3]
```

```
B =
```

```
2 2
```

```
3 3
```

```
>> C=A*B
```

```
C =
```

```
8 8
```

```
18 18
```

Multiplicative operations- element-by-element

```
>> C1=A.*B
```

```
C1 =
```

```
2 4
```

```
9 12
```

Matrix operations

We can do matrix arithmetic using the standard arithmetical operators.

Division operations - based on definition

```
>> A=[1 2;3 4];
```

```
>> B=[2 2; 3 1];
```

- $A \setminus B$ backslash or left matrix divide

It is the same as $A^{-1} * B$

```
>> C=A\B
```

C =

```
-1.0000 -3.0000
```

```
1.5000 2.5000
```

- A/B Slash or right matrix divide.

It is the same as $A * B^{-1}$

```
>> C1=A/B
```

C1 =

```
1.2500 -0.5000
```

```
2.2500 -0.5000
```

Division operations- element-by-element

```
>> C2=A.\B
```

C2 =

```
2.0000 1.0000
```

```
1.0000 0.2500
```

```
>> C3=A./B
```

C3 =

```
0.5000 1.0000
```

```
1.0000 4.0000
```


Basic array information.

1. **Size - Array dimensions**

returns the sizes of each dimension of array X in a vector or matrix.

```
>> d = size(X)
```

```
>> [m,n] = size(X)
```

2. **Length - Length of largest array dimension**

returns the length of the largest array dimension in X.

```
>>L = length(X)
```

3. **Isempty - Determine whether array is empty**

returns logical 1 (true) if A is an empty array and logical 0 (false) otherwise.

```
>>TF = isempty(A)
```

Basic array information.

4. Max - Largest elements in array

returns the largest elements of A.

```
>>M = max(A)
```

5. Min - Smallest elements in array

returns the smallest elements of A.

```
>> M = min(A)
```

6. Sum - Sum of array elements

returns the sum of the elements of A along the first array dimension whose size does not equal 1.

```
>> S = sum(A)
```

7. Prod - Product of array elements

returns the product of the array elements of A.

```
>> B = prod(A)
```

Matrix functions - numerical linear algebra.

1. Rank - Rank of matrix

returns the number of singular values of A

```
>> k = rank(A)
```

2. Det - Matrix determinant

returns the determinant of square matrix A.

```
>> d = det(A)
```

3. Trace - Sum of diagonal elements

is the sum of the diagonal elements of the matrix A.

```
>> b = trace(A)
```

4. Inv - Matrix inverse

returns the inverse of the square matrix X.

```
>> Y = inv(X)
```

Matrix functions - numerical linear algebra.

5. Eig - Eigenvalues and eigenvectors

returns a column vector containing the eigenvalues of square matrix A.

```
>> e = eig(A)
```

6. Poly - Polynomial with specified roots

This MATLAB function where A is an n-by-n matrix returns an n+1 element row vector whose elements are the coefficients of the characteristic polynomial $\rightarrow \det(\lambda I - A)$.

```
>> p = poly(A)
```

7. Matrix transposed

```
>> A_transposed = A'
```

Polynomials

Representing polynomials:

Matlab software represents polynomials as row vectors containing coefficients ordered by descending powers. For example, consider the equation:

$$p(x) = x^3 - 2x - 5$$

To enter this polynomial into MATLAB, use:

```
>> p=[1 0 -2 -5]
```

```
p =
```

```
    1    0   -2   -5
```

A coefficient of 0 indicates an intermediate power that is not present in the equation.

Evaluating polynomials

Polyval – the function evaluates polynomial at specified value. To evaluate p at p=5, use

```
>> polyval(p,5)
```

```
ans =
```

```
    110
```

Polynomial derivative

Polyder - the function calculates the derivative of polynomials, polynomial products, and polynomial quotients. The operands a, b, and p are vectors whose elements are the coefficients of a polynomial in descending powers.

$k = \text{polyder}(p)$ returns the derivative of the polynomial p.

$k = \text{polyder}(a,b)$ returns the derivative of the product of the polynomials a and b.

Example: a) $p1(x) = x^3 - 2x - 5$; b) $p2(x) = (3x^2 + 6x + 9)(x^2 + 2x)$

a)

```
>> p1=[1 0 -2 -5];
```

```
>> polyder(p1)
```

```
ans =
```

```
3    0   -2
```

b)

```
>> a=[3 6 9];
```

```
>> b=[1 2 0];
```

```
>> polyder(a,b)
```

```
ans =
```

```
12   36   42   18
```

This result represents the polynomial: $12x^3 + 36x^2 + 42x + 18$

Polynomial roots

Roots – the function solves polynomial equations of the form $a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0 = 0$

Polynomial equations contain a single variable with nonnegative exponents.

This function returns the roots of the polynomial represented by p as a column vector.

Example: $p = x^3 - 2x - 5$

```
>> p=[1 0 -2 -5];
```

```
>> roots(p)
```

```
ans =
```

```
2.0946 + 0.0000i
```

```
-1.0473 + 1.1359i
```

```
-1.0473 - 1.1359i
```

Residue - Convert between partial fraction expansion and ratio of two polynomials

`[r,p,k] = residue(b,a)` finds the residues, poles, and direct term of a Partial Fraction Expansion of the ratio of two polynomials, where the expansion is of the form

$$\frac{b(p)}{a(p)} = \frac{b_m p^m + b_{m-1} p^{m-1} + \dots + b_1 p + b_0}{a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0}$$

The inputs to `residue` are vectors of coefficients of the polynomials:

$b = [b_m \ b_{m-1} \ \dots \ b_1 \ b_0]$ and $a = [a_n \ a_{n-1} \ \dots \ a_1 \ a_0]$

The outputs are the residues $r = [r_n \ r_{n-1} \ \dots \ r_1]$, the poles $p = [p_n \ p_{n-1} \ \dots \ p_1]$, and the polynomial k .

```
>> b=1;
```

```
>> a=[1 5 6];
```

```
>> [r,p,k]=residue(b,a)
```

```
r =
```

```
-1.0000
```

```
1.0000
```

```
p =
```

```
-3.0000
```

```
-2.0000
```

```
k =
```

```
[]
```


[b,a] = residue(r,p,k) converts the partial fraction expansion back to the ratio of two polynomials and returns the coefficients in b and a.

Conv - Convolution and polynomial multiplication

returns the convolution of vectors u and v.

Example: $(x + 2)(2x^2 + 4x + 1) = 2x^3 + 8x^2 + 9x + 2$

```
>> u=[1 2];
```

```
>> v=[2 4 1]
```

```
v =
```

```
    2    4    1
```

```
>> conv(u,v)
```

```
ans =
```

```
    2    8    9    2
```

General information

Managing the workspace

- `who` - List variables in workspace
- `whos` - List variables in workspace, with sizes and types
- `clear` - Remove items from workspace, freeing up system memory

```
>> clear all
```

```
>> clear name1 ... nameN
```

- `save` - Save workspace variables to file

This function saves all variables from the current workspace in a MATLAB formatted binary file (MAT-file) called `filename`.

```
>> save(filename)
```

```
>> save(filename,variables)
```

- `load` - Load variables from file into workspace

```
>> load(filename)
```

```
>> load(filename,variables)
```

Special values

- **Ans** - variable automatically when you specify no output argument.
- **Pi** - returns the floating-point number nearest the value of π . : 3.1415926535897...
- **NaN** - Not-a-Number. NaN returns the IEEE® arithmetic representation for Not-a-Number (NaN). These values result from operations which have undefined numerical results.

These operations produce NaN:

- Any arithmetic operation on a NaN, such as `sqrt(NaN)`
- Addition or subtraction, such as magnitude subtraction of infinities as `(+Inf)+(-Inf)`
- Multiplication, such as `0*Inf`
- Division, such as `0/0` and `Inf/Inf`
- **Inf** – Infinity - returns the arithmetic representation for positive infinity. Infinity values result from operations like division by zero and overflow, which lead to results too large to represent as conventional floating-point values.
- **i, j** - Imaginary unit.

References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.

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External Description of Dynamic Systems

External description of dynamic systems

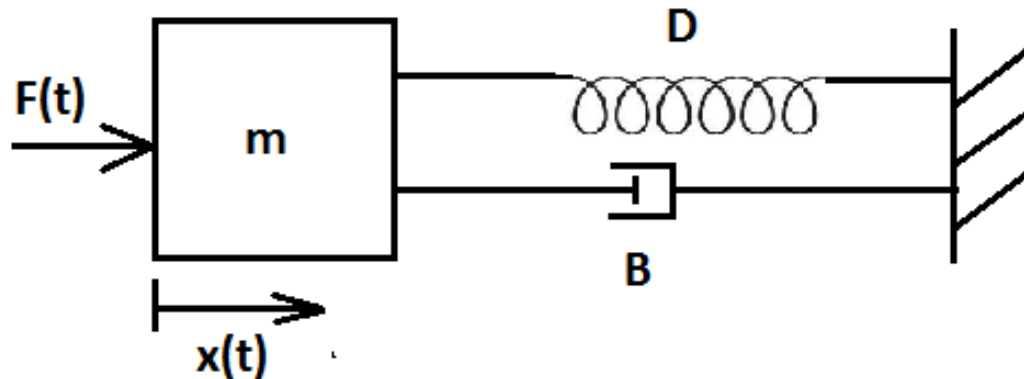
- Differential equation
- Transfer function (time constants, poles and zeros of transfer function)
- Impulse response
- Step response (relation between responses)
- P-Z map (pole-zeros configuration)
- Frequency response
 1. Nyquist plot of transfer function
 2. Bode plot of transfer function

Differential equation

- To develop differential equations for linear continuous-time systems, if we do not know the internal structure, we apply an input (a step or an impulse) and measure its response. Then we try to develop a differential equation from the measured data.
- In the second approach, we must know the internal structure of a system. Then we can apply physical laws, such as Newton's and Kirchhoff's laws. This approach can be illustrated by examples.

Mechanical system

- System consists of a block with mass m connected to a wall by a spring (D) and a dashpot which represents the shock absorber (B). The input is an external force $F(t)$ applied to the block, and the output is the distance $x(t)$ measured from the equilibrium position. If $F(t) = 0$ then $x(t) = 0$.
- Strictly speaking the spring and dashpot is nonlinear, but it can be modeled as linear element with spring constant D and dashpot constant B . The spring generates the force $F_D(t)$ proportional to $D \cdot x(t)$, the dashpot generates a viscous friction $F_B(t)$ proportional to $B \cdot x'(t)$. The Newton law of inertia of a mass is $F_m = m \cdot x''(t)$.



Mechanical system

To move the block we must overcome the spring force, friction force and force of inertia.
Therefore we have:

$$F(t) = F_m(t) + F_B(t) + F_D(t) = m \frac{d^2x}{dt^2} + B \frac{dx}{dt} + Dx = m\ddot{x}(t) + B\dot{x}(t) + Dx(t)$$

$F(t)$ external force, the input of system

$F_m(t)$ force implies acceleration, law of inertia

$F_D(t)$ spring force

$F_B(t)$ viscous friction force

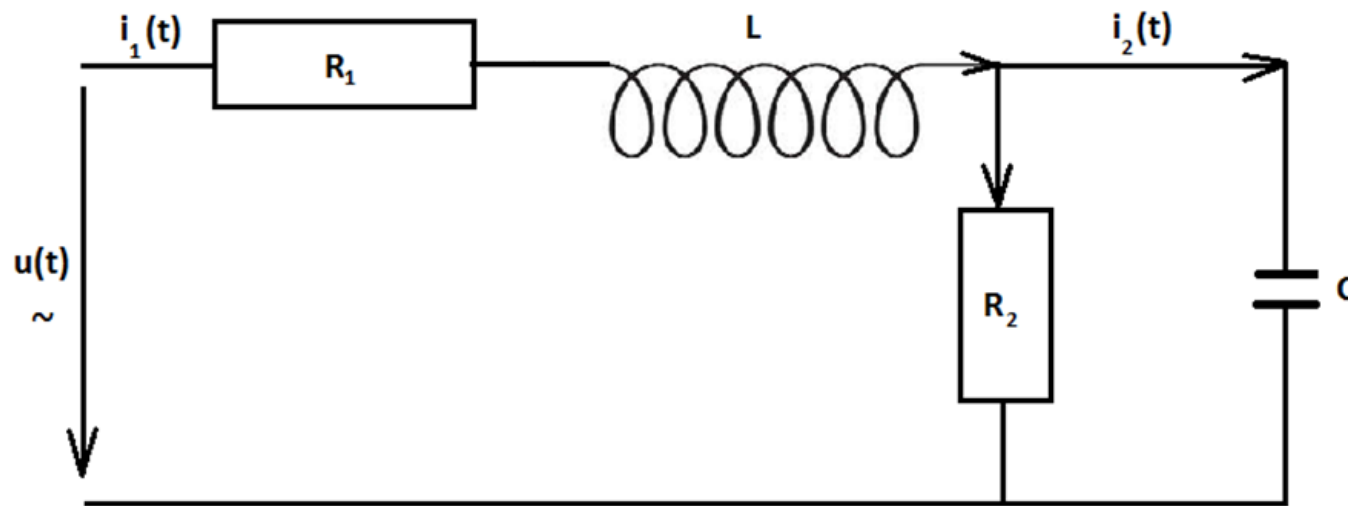
$x(t)$ distance from equilibrium position, the output of system

Description of dynamic behavior of systems can be often approached by ordinary differential equation with time as independent variable. Linear system is described by linear differential equation.

Electrical circuit

Set up differential equations of electrical circuit. Consider the voltage $u(t)$ as input and current $i_1(t)$ as output, where R_1 is resistance, L inductance and C capacitance.

R_1 , R_2 , L , C are constants.



Electrical circuit

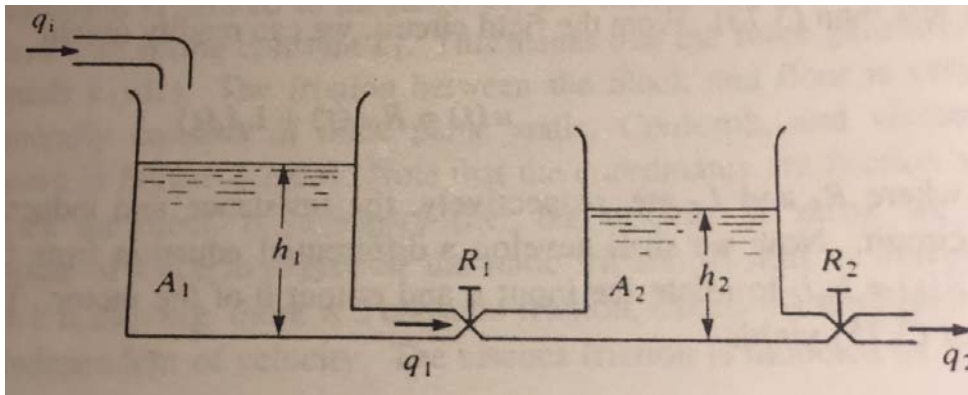
We use Kirchhoff's voltage law to develop a differential equations of the network. (The directed sum of the electrical potential around the loop must be zero.)

$$\text{Loop I: } u(t) = i_1(t) \cdot R_1 + L \frac{di_1}{dt} + R_2 (i_1(t) - i_2(t))$$

$$\text{Loop II: } 0 = \frac{1}{C} \int_0^t i_2(t) dt + R_2 (i_2(t) - i_1(t))$$

Hydraulic tanks

In chemical plants, it is often necessary to maintain the levels of liquids. A simplified model is shown in figure:



q_i , q_1 , q_2 = rates of the flow of liquid

A_1 , A_2 = areas of the cross-section of tanks

h_1 , h_2 = liquid levels

R_1 , R_2 = flow resistance, controlled by valves

It is assumed that q_1 and q_2 are governed by

$$q_1 = \frac{h_1 - h_2}{R_1} \quad \text{and} \quad q_2 = \frac{h_2}{R_2}$$

Hydraulic tanks

The change of liquid are governed by

$$A_1 dh_1 = (q_i - q_1) dt$$

$$A_2 dh_2 = (q_1 - q_2) dt$$

Or

$$A_1 h_1'(t) = (q_i - q_1)$$

$$A_2 h_2'(t) = (q_1 - q_2)$$

$$h_1'(t) = \frac{(q_i - q_1)}{A_1}$$

$$h_2'(t) = \frac{(q_1 - q_2)}{A_2}$$

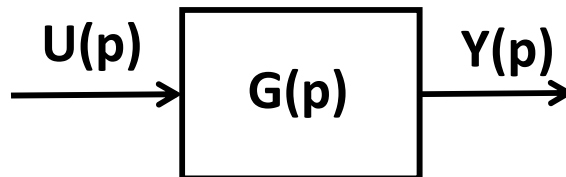
Transfer function

Transfer function is a mathematical representation of the relation between the input and output of a linear time-invariant system.

Definition formula of transfer function:

$$G(p) = \frac{Y(p)}{U(p)} = \frac{L \{ y(t) \}}{L \{ x(t) \}} = \frac{\text{output}}{\text{input}}$$

The transfer function $G(p)$ of a control system is defined as the ratio of the Laplace transform of the output variable $Y(p)$ to the Laplace transform of the input variable $U(p)$ assuming all initial conditions as zero.



Transfer function of mechanical system

In our case the input signal is the change of external force $F(t)$ and the response (output) is the change of the position of the block $x(t)$. Assuming the initial conditions as zero and taking Laplace transform on both sides of equation (1), the following equations are obtained:

$$F(t) = m \frac{d^2 x}{dt^2} + B \frac{dx}{dt} + Dx \quad (1)$$

$$F(p) = m \cdot p^2 X(p) + B \cdot pX(p) + D \cdot X(p)$$

$$F(p) = X(p) \cdot (m \cdot p^2 + B \cdot p + D)$$

$$G(p) = \frac{X(p)}{F(p)} = \frac{1}{m \cdot p^2 + B \cdot p + D}$$

Transfer function of electrical system

Consider the voltage $u(t)$ as input and current $i_1(t)$ as output. Taking Laplace transform on both sides of equation (2) and (3), the following equations are obtained:

$$u(t) = i_1(t) \cdot R_1 + L \frac{di_1}{dt} + R_2 (i_1(t) - i_2(t)) \quad (2)$$

$$0 = \frac{1}{C} \int_0^t i_2(t) dt + R_2 (i_2(t) - i_1(t)) \quad (3)$$

$$U(p) = I_1(p) \cdot R_1 + L \cdot p \cdot I_1(p) + R_2 (I_1(p) - I_2(p))$$

$$0 = \frac{1}{p \cdot C} I_2(p) + R_2 (I_2(p) - I_1(p))$$

Transfer function:
$$G(p) = \frac{I_1(p)}{U(p)} = \frac{p \cdot CR_2 + 1}{p^2 \cdot LR_2C + p \cdot (R_1R_2C + L) + R_1 + R_2}$$

Poles and zeros of transfer function

We can now determine the transfer function more generally.

For a differential equation:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = b_m u^{(m)} + b_{m-1} u^{(m-1)} + \dots + b_1 u' + b_0 u$$

With zero initial condition $y^{(n-1)} = y^{(n-2)} = y = 0$ and $u^{(n-1)} = u^{(n-2)} = u = 0$ at $t = 0$.

When y is output and u is input, the corresponding transform function can be expressed in the form of a quotient of polynomials in the following form:

$$G(p) = \frac{Y(p)}{U(p)} = \frac{b_m p^m + b_{m-1} p^{m-1} + \dots + b_1 p + b_0}{a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0} = \frac{B(p)}{A(p)}$$

Where $A(p)$ is called characteristic polynomial.

For physically realistic processes the highest power of p in the numerator is either equal or less than that of the denominator: $n \geq m$

Poles and zeros of transfer function

Conclusion: $G(p)$ contains information of the dynamic behavior of a model as represented by the differential equation.

Denominator of $G(p)$ is called the **characteristic polynomial** of the diff. equation.

The roots of the characteristic equation $A(p) = 0$ are called the **poles** of $G(p)$.

The roots of the numerator of $G(p)$: $B(p) = 0$ are called the **zeros** of $G(p)$.

The graphical symbol for pole is X and for zero O. The said symbols are used when poles and zeros are to be shown on a real and imaginary axes (p -plane)

Transfer function in pole-zero form

We can factor the transfer function into so-called **pole-zero form**.

$$G(p) = \frac{B(p)}{A(p)} = \left(\frac{b_m}{a_n} \right) \frac{(p - z_1)(p - z_2) \dots (p - z_m)}{(p - p_1)(p - p_2) \dots (p - p_n)}$$

When zeros are denoted z_1, z_2, \dots, z_m

poles are denoted p_1, p_2, \dots, p_n

If all roots are real, we can factor the transfer function into so-called **time-constant form**.

$$G(p) = \frac{Q(p)}{P(p)} = \left(\frac{b_0}{a_0} \right) \frac{(\tau_a p + 1)(\tau_b p + 1) \dots (\tau_m p + 1)}{(\tau_1 p + 1)(\tau_2 p + 1) \dots (\tau_n p + 1)}$$

Where $K = \left(\frac{b_0}{a_0} \right)$ is known as the gain factor of transfer function.

τ_n are called the time constants of the system

We can also use the time-constant notation for express the poles:

$$p_1 = -\frac{1}{\tau_1}, \quad p_2 = -\frac{1}{\tau_2}, \quad p_n = -\frac{1}{\tau_n}$$

The **poles determine** qualitatively the **dynamic behavior** of the model of differential equitation.

The term “roots of the characteristic equation” is used interchangeably with “poles of transfer function”.

Conclusion: The transfer function of a system is completely specified in term of its poles and zeros and the gain factor.

Procedure for determining the transfer function of a control system.

- Formulate the differential equations for the system.
- Take the L-transform of the system equations, assuming the initial conditions as zero.
- Specify the system output and the input.
- Take the ratio of the Laplace transform of the output and the Laplace transform of the input. The obtained ratio is the required transfer function.

Problems

For the transfer function plot the poles and zeros in p -plane.

$$G(p) = \frac{1}{2} \frac{(p^2 + 4)(1 + 2,5 p)}{(p^2 + 2)(1 + 0,5 p)}$$

2. The dynamic system is done by differential equation. Find the transfer function $G(p)$.

$$5y'' - 2y' + 3y = u'' + 4u' + 4u$$

3. Find zeros and poles of dynamic system which are done by transfer function. Plot it in p -plane.

$$G(p) = \frac{p^2 + 4p + 4}{5p^3 - 2p + 3}$$

Responses of the system

1. **Impulse response** – is the inverse Laplace transform of the output $g(t)$, if the input is specified as unit impulse (Dirac impulse) $\delta(t)$

$$u(t) = \delta(t) \Rightarrow U(p) = 1$$

$$G(p) = \frac{Y(p)}{U(p)} \Rightarrow Y(p) = G(p)U(p) = G(p) \cdot 1$$

$$Y(p) = G(p)$$

$$g(t) = L^{-1}\{G(p)\}$$

2. **Step response** – is the inverse Laplace transform of the output $h(t)$, if the input is specified as unit step (Heaviside function)

$$u(t) = 1(t) \Rightarrow U(p) = \frac{1}{p}$$

$$G(p) = \frac{Y(p)}{U(p)} \Rightarrow Y(p) = H(p) = G(p)U(p) = G(p) \cdot \frac{1}{p}$$

$$y(t) = h(t) = L^{-1}\left\{G(p)\frac{1}{p}\right\}$$

Responses of the system

4. Response, if the input is **ramp function** $u(t)=t$.

$$u(t) = t \Rightarrow U(p) = \frac{1}{p^2}$$

$$G(p) = \frac{Y(p)}{U(p)} \Rightarrow Y(p) = G(p)U(p) = G(p) \cdot \frac{1}{p^2}$$

$$y(t) = L^{-1} \left\{ G(p) \frac{1}{p^2} \right\}$$

5. Response, if the input is **harmonic function**.

$$u(t) = A \cdot \sin \omega t \Rightarrow U(p) = A \frac{\omega}{p^2 + \omega^2}$$

$$G(p) = \frac{Y(p)}{U(p)} \Rightarrow Y(p) = G(p)U(p) = G(p) \cdot A \frac{\omega}{p^2 + \omega^2}$$

$$y(t) = L^{-1} \left\{ G(p) \cdot A \frac{\omega}{p^2 + \omega^2} \right\}$$

Solved example

System has the transfer function: $G(p) = \frac{1}{p^2 + 5p + 6}$

1. Compute the impulse response $g(t)$ in time domain.
2. Compute the step response $H(p)$ and in time domain $h(t)$.
3. Compute steady-state value (final value) for both responses using the limits theorems.
4. Compute step response $h(t)$ from $g(t)$ and impulse response $g(t)$ from $h(t)$.
5. Simulate the response using Matlab and Matlab-Simulink.

1. Impulse response:

$$g(t) = L^{-1}\{G(p)\} = L^{-1}\left\{\frac{1}{p^2 + 5p + 6}\right\} = L^{-1}\left\{\frac{1}{(p+3)(p+2)}\right\} = L^{-1}\left\{\frac{-1}{(p+3)} + \frac{1}{(p+2)}\right\} = -e^{-3t} + e^{-2t}$$

2. Step response:

$$h(t) = L^{-1}\left\{G(p) \cdot \frac{1}{p}\right\} = L^{-1}\{H(p)\} = L^{-1}\left\{\frac{1}{(p^2 + 5p + 6)p}\right\} = L^{-1}\left\{\frac{\frac{1}{3}}{(p+3)} + \frac{-\frac{1}{2}}{(p+2)} + \frac{\frac{1}{6}}{p}\right\} = \frac{1}{3}e^{-3t} - \frac{1}{2}e^{-2t} + \frac{1}{6}$$

3. Final value of the impulse response

$$g(t \rightarrow \infty) = \lim_{p \rightarrow 0} p \cdot G(p) = \lim_{p \rightarrow 0} p \frac{1}{p^2 + 5p + 6} = 0$$

Verification in time domain: $\lim_{p \rightarrow \infty} g(t) = 0$

4. Final value of the step response

$$h(t \rightarrow \infty) = \lim_{p \rightarrow 0} p \cdot H(p) = \lim_{p \rightarrow 0} p \frac{1}{(p^2 + 5p + 6)p} = \frac{1}{6}$$

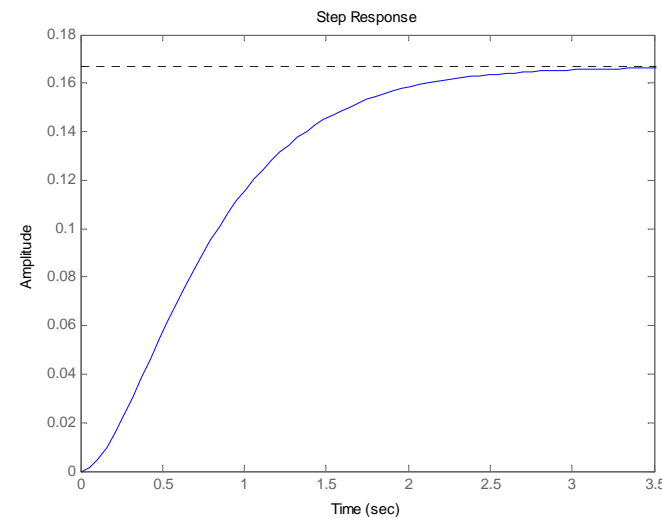
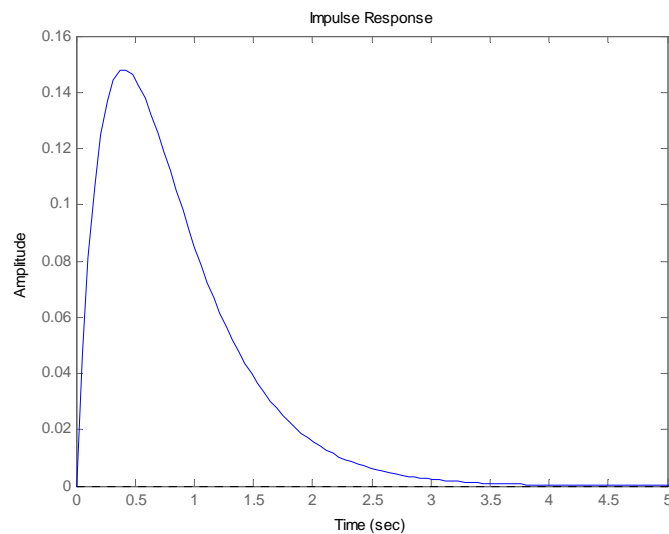
Verification in time domain: $\lim_{p \rightarrow \infty} h(t) = \frac{1}{6}$

5. Relation between impulse response and step response:

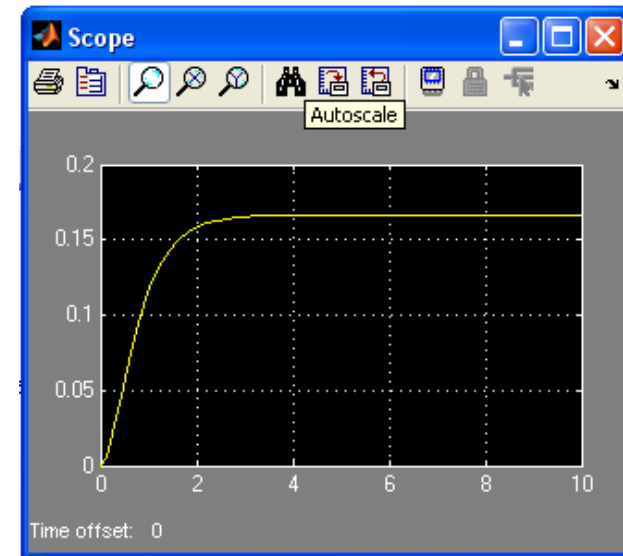
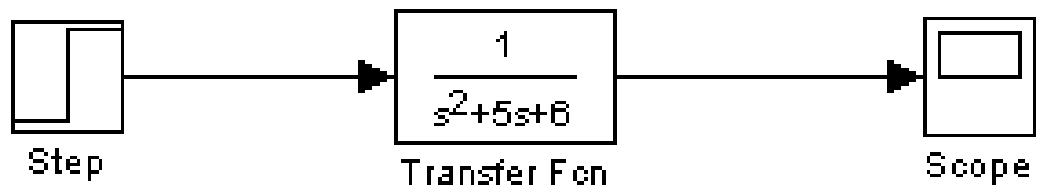
$$h(t) = \int_0^t g(t) dt = \int_0^t \left(-e^{-3t} + e^{-2t} \right) dt = \left[\frac{1}{3} - e^{-3t} - \frac{1}{2} e^{-2t} \right]_0^t = \frac{1}{3} - e^{-3t} - \frac{1}{2} e^{-2t} - \frac{1}{3} + \frac{1}{2}$$

$$g(t) = \frac{d h(t)}{d t} = \frac{d}{d t} \left(\frac{1}{3} - e^{-3t} - \frac{1}{2} e^{-2t} + \frac{1}{6} \right) = e^{-3t} - e^{-2t}$$

6. Simulation using Matlab.



Simulation of the step response using Simulink:



Frequency response – Nyquist plot

Procedure for mapping $G(p)$ from p -plane to $G(j\omega)$ -plane.

For plotting $G(j\omega)$ the independent variable p is varied on the entire imaginary axis from $\omega=+0$ to $\omega=+\infty$.

The plot variation from $\omega = -0$ to $\omega = -\infty$ use vertical image of the plot from $\omega = +0$ to $\omega = +\infty$.

The examples given below illustrates the process of Nyquist plot.

$$G(p) = \frac{100}{p^2 + p + 20}$$

$$G(j\omega) = \frac{100}{(j\omega)^2 + j\omega + 20}$$

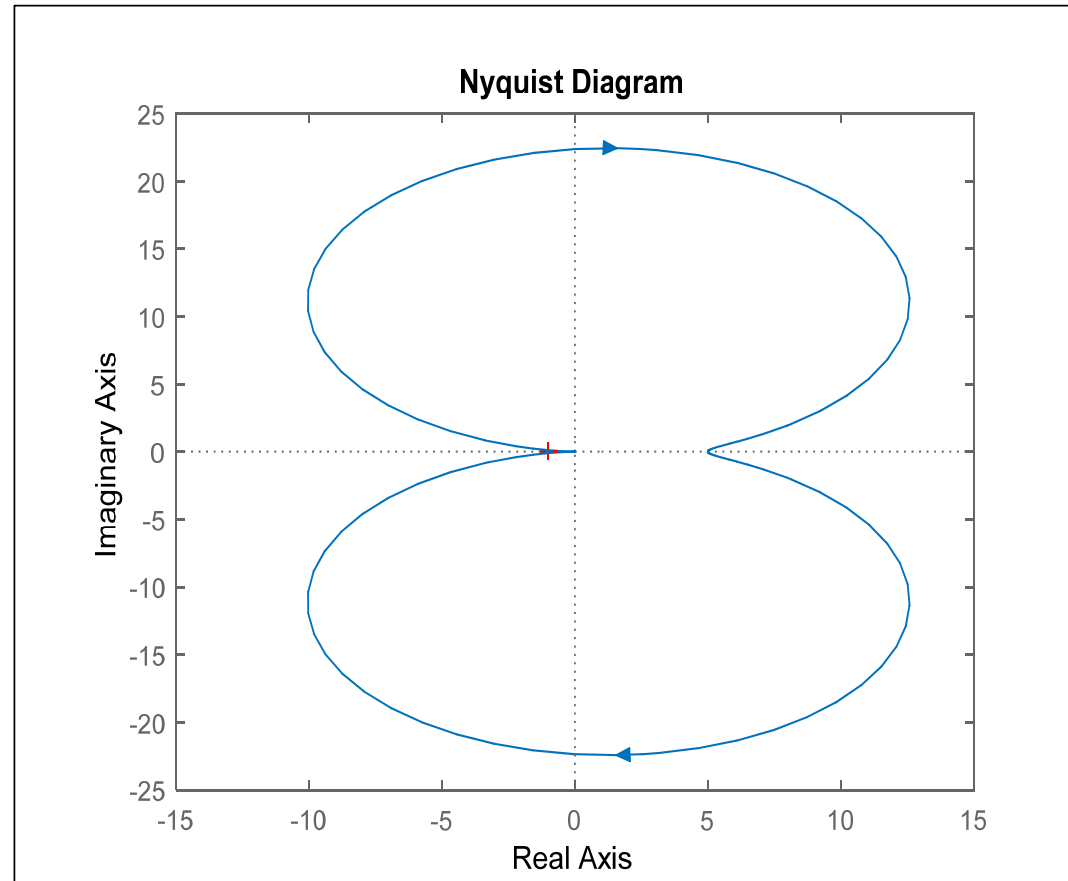
$$\begin{aligned} G(j\omega) &= \frac{100}{(j\omega)^2 + j\omega + 20} = \frac{100}{-\omega^2 + j\omega + 20} = \frac{100}{(20 - \omega^2) + j\omega} = \frac{100}{(20 - \omega^2) + j\omega} \cdot \frac{(20 - \omega^2) - j\omega}{(20 - \omega^2) - j\omega} \\ &= \frac{100(20 - \omega^2) - 100j\omega}{400 - 39\omega^2 + \omega^4} \end{aligned}$$

$$\operatorname{Re}\{G(j\omega)\} = \frac{(20 - \omega^2)}{400 - 39\omega^2 + \omega^4} \cdot 100$$

$$\operatorname{Im}\{G(j\omega)\} = \frac{-\omega}{400 - 39\omega^2 + \omega^4} \cdot 100$$

Frequency response – Nyquist plot

ω	$Re\{G(j\omega)\}$	$Im\{G(j\omega)\}$
0	5	0
1	5.25	-0.28
2	6.15	-0.77
3	8.46	-2.31
4	12.5	-12.5
$\sqrt{20}$	0	-22
5	-10	-10
6	-5.48	-2.05
8	-2.2	-0.4
10	-1.23	-0.15
∞	0	0



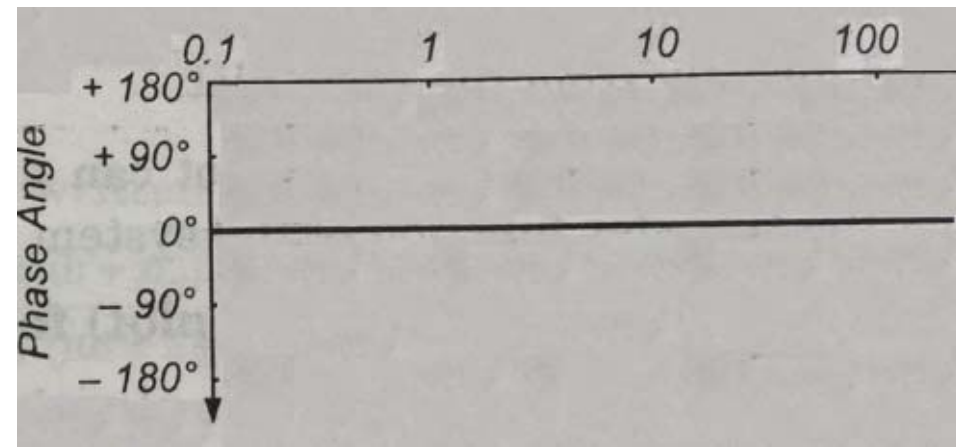
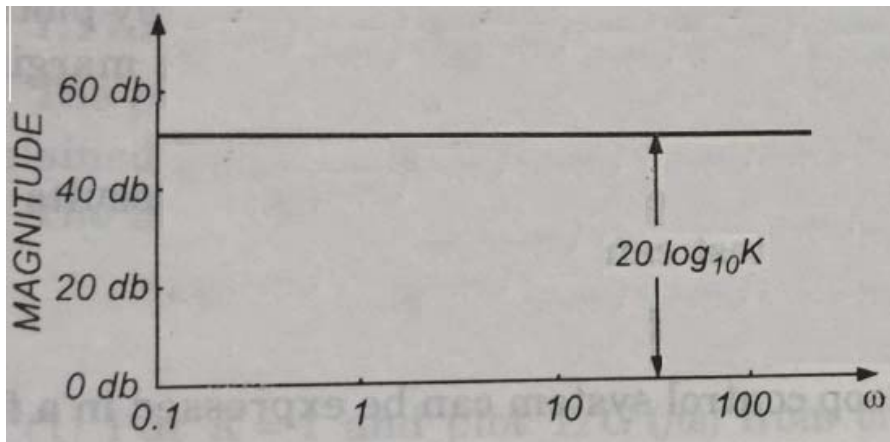
Frequency response: Bode plot (Logarithmic plot)

Bode plot is a graph consisting of two parts:

- a) Magnitude of $G(j\omega)$ in decibel ie. $20\log_{10}|G(j\omega)|$ versus $\log_{10}\omega$
- b) Phase angle of $G(j\omega)$ versus $\log_{10}\omega$

1. Graphs for the gain term K

- a) Magnitude: $K(\text{db}) = 20 \log_{10} K$
- b) Phase angle is always zero: $\varphi(\omega) = 0$

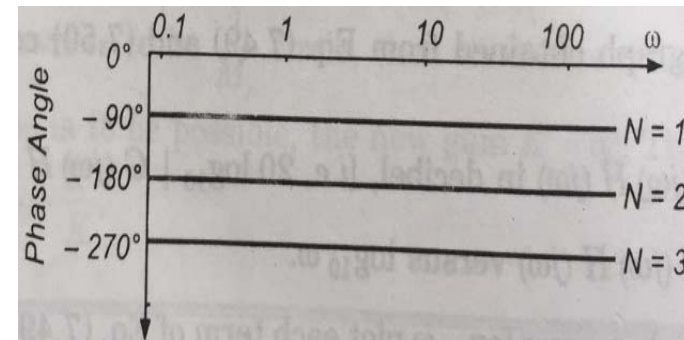
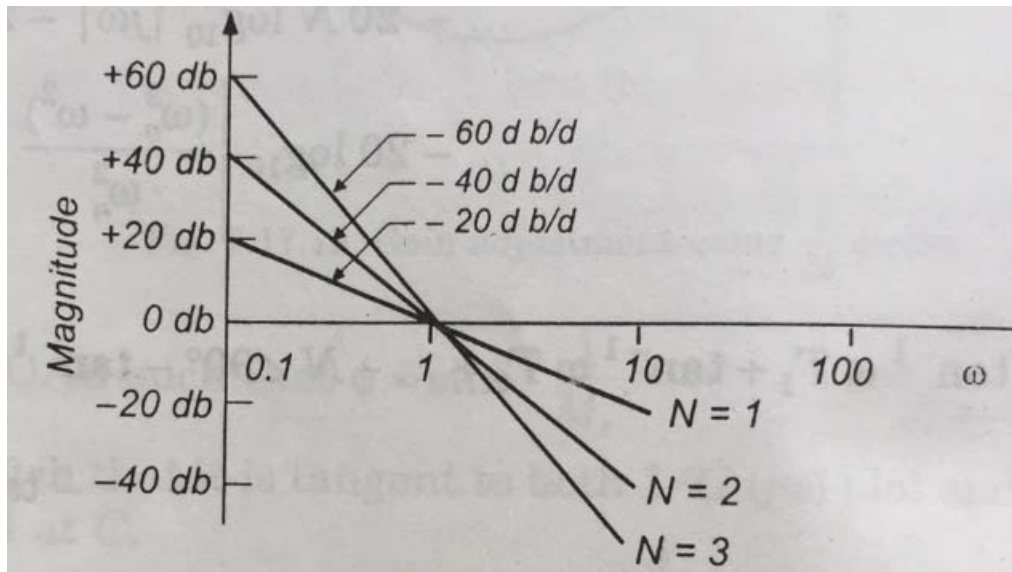


Frequency response: Bode plot (Logarithmic plot)

2) Graphs for the term $G(j\omega) = \frac{1}{(j\omega)^N}$

a) Magnitude in decibel $A(db) = -20N \log_{10} \omega$

b) Phase angle: $\varphi(\omega) = -90^\circ$



Frequency response: Bode plot (Logarithmic plot)

3) Graphs for the term $G(j\omega) = (1 + j\omega T)$

a) The magnitude in decibel is given: $A(db) = 20 \log_{10}|(1 + j\omega T)| = 20 \log_{10} \sqrt{1 + \omega^2 T^2}$

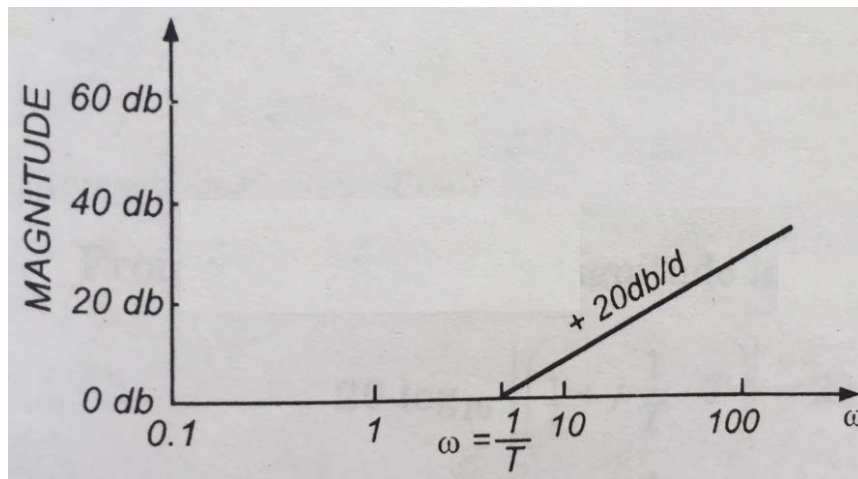
Consider following two cases:

i) $\omega T \ll 1$, ωT is negligible as compared to 1: $A(db) = 20 \log_{10}|(1 + j\omega T)| \cong 20 \log_{10} 1 = 0 \text{ db}$

ii) $\omega T \gg 1$, 1 is negligible as compared to ωT :

$$A(db) = 20 \log_{10}|(1 + j\omega T)| \cong 20 \log_{10} \sqrt{\omega^2 T^2} = 20 \log_{10} \omega T = 20 \log_{10} \omega - 20 \log_{10}(1/T)$$

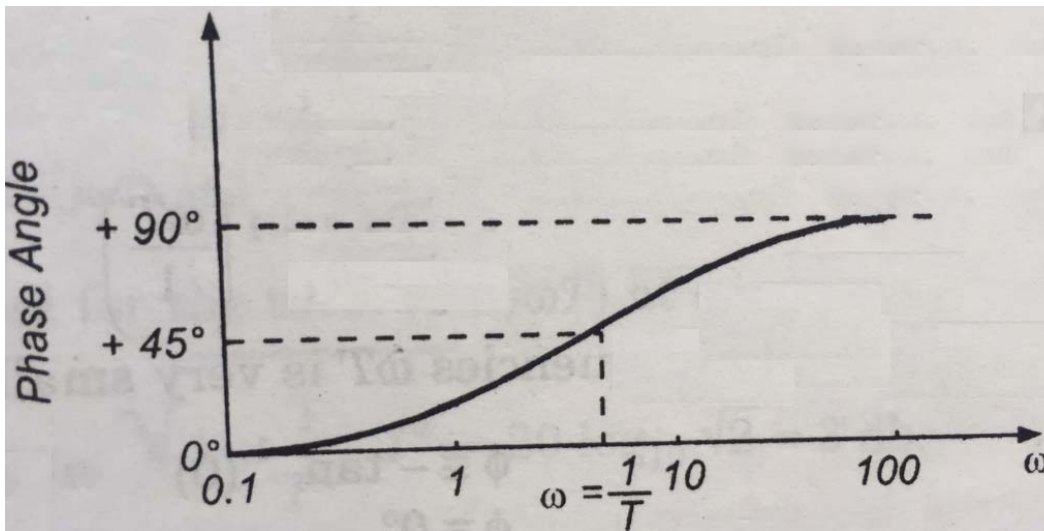
Hence, the two graphs intersect on 0db axis at $\omega = 1/T$, for case ii) the graph has a slope of 20db/decade.



Frequency response: Bode plot (Logarithmic plot)

b) The phase angle: $\varphi(\omega) = \arctan\left(\frac{\omega T}{1}\right)$

- i) At very low frequencies ωT is very small: $\varphi = \arctan(0) = 0^\circ$
- ii) At $\omega = 1/T$: $\varphi = \arctan(1/T \cdot 1) = \arctan(1) = 45^\circ$
- iii) At very high frequencies ωT is very large: $\varphi = \arctan(\infty) = 90^\circ$



Frequency response: Bode plot (Logarithmic plot)

4. Graphs for term $G(j\omega) = \frac{1}{(1 + j\omega T)}$

a) The magnitude in decibel:

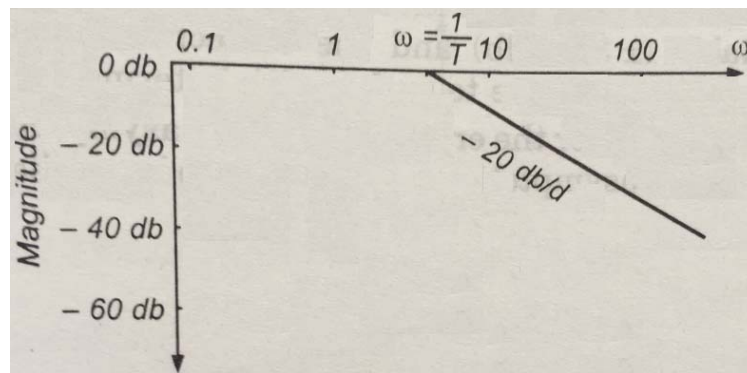
Consider following two cases: $A(db) = 20 \log_{10} \left| \frac{1}{(1 + j\omega T)} \right| = 20 \log_{10} \left(\frac{1}{\sqrt{1 + \omega^2 T^2}} \right) = -20 \log_{10} \sqrt{1 + \omega^2 T^2}$

i) $\omega T \ll 1$, ωT is negligible as compared to 1: $A(db) = 20 \log_{10} \left| \frac{1}{(1 + j\omega T)} \right| \cong -20 \log_{10} 1 = 0 \text{ db}$

ii) $\omega T \gg 1$, 1 is negligible as compared to ωT :

$$A(db) = 20 \log_{10} \left| \frac{1}{(1 + j\omega T)} \right| \cong -20 \log_{10} \sqrt{\omega^2 T^2} = -20 \log_{10} \omega T = -20 \log_{10} \omega + 20 \log_{10} \left(\frac{1}{T} \right)$$

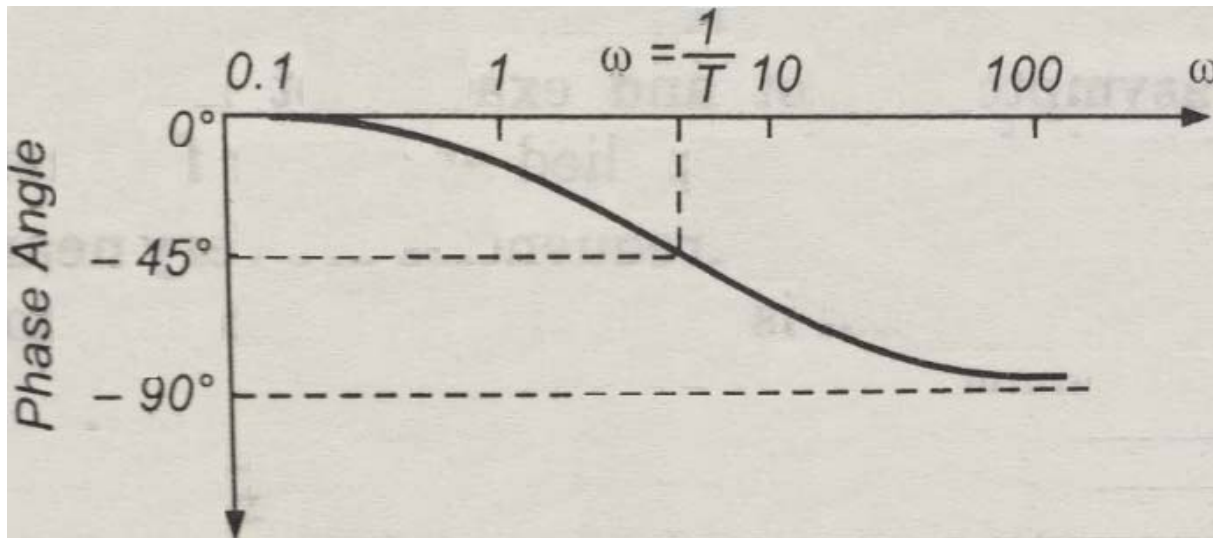
Hence, the two graphs intersect on 0db axis at $\omega = 1/T$, for case ii) the graph has a slope of -20db/decade.



Frequency response: Bode plot (Logarithmic plot)

b) The phase angle: $\varphi(\omega) = -\arctan\left(\frac{\omega T}{1}\right)$

- i) At very low frequencies ωT is very small: $\varphi = -\arctan(0) = 0^\circ$
- ii) At $\omega=1/T$: $\varphi=\arctan(1/T \cdot 1)=\arctan(1) = -45^\circ$
- iii) At very high frequencies ωT is very large: $\varphi=\arctan(\infty) = -90^\circ$

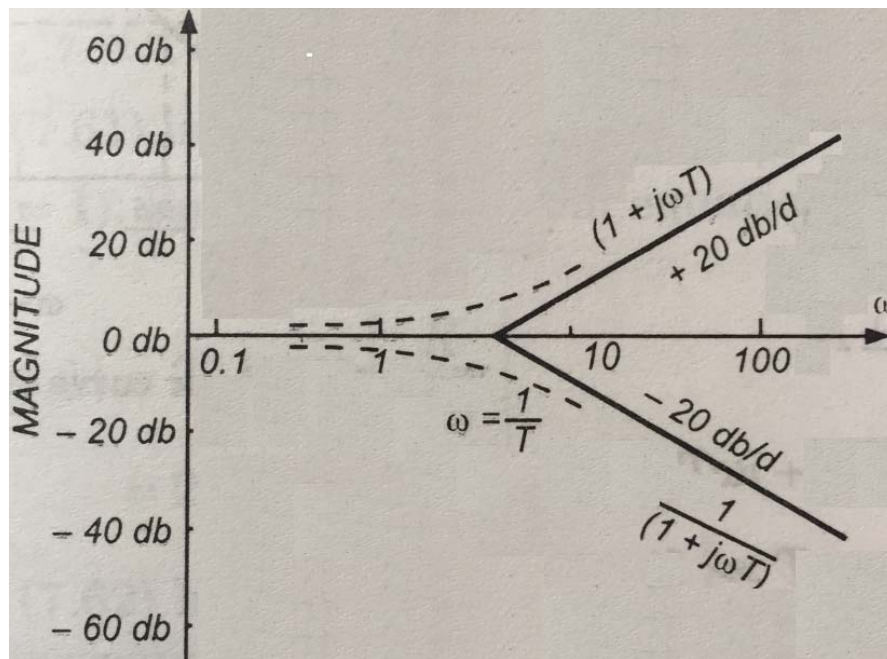


Frequency response: Bode plot (Logarithmic plot)

For exact plot the magnitude in decibel for the term $(1+j\omega T)$ at $\omega=1/T$ is 3db. Similarly the maximum error for the term $1/(1+j\omega T)$ is -3db. An error of ± 3 db is permissible.

Exact and asymptotic (approximate) plots for the terms $G(j\omega) = (1 + j\omega T)$ and

$$G(j\omega) = \frac{1}{(1 + j\omega T)}$$



Bode plot: Solved example

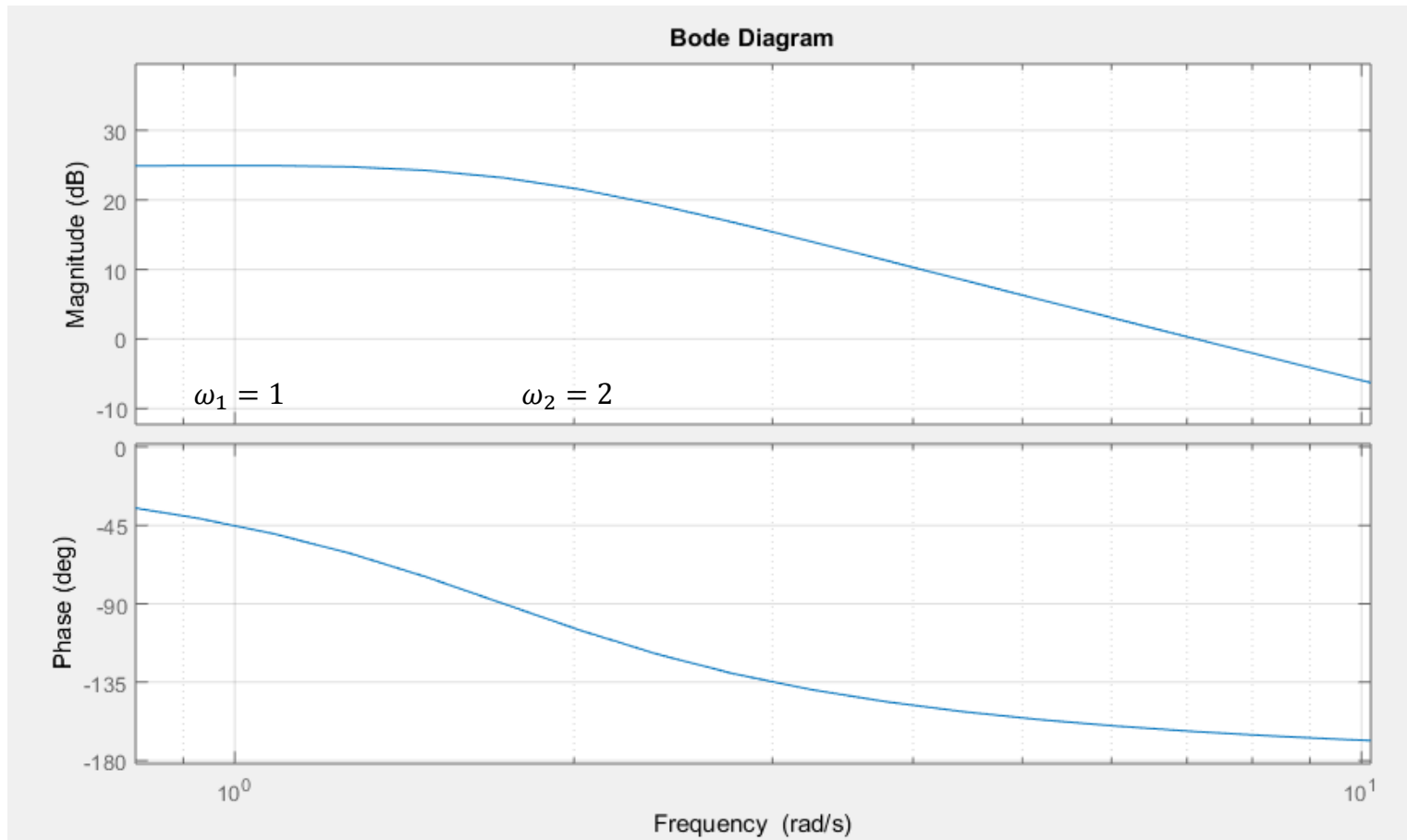
Sketch the Bode plot for transfer function: $G(p) = \frac{50}{(p+1)(p+2)}$

Solution:

- put $p=j\omega$ and rewrite the transfer function in Bode form: $G(j\omega) = \frac{50/2}{(j\omega+1)(0.5j\omega+1)}$
- Time constants: $T_1 = 1 \quad T_2 = 0.5$
- Magnitude in decibel: $|G(j\omega)|_{db} = 20 \log_{10} 25 - 20 \log_{10} |j\omega+1| - 20 \log_{10} |0.5j\omega+1|$
- Initial magnitude: $A_{db} = 20 \log_{10} 25 = 27,9$
- Corner frequency of the asymptotic plot: $\omega_1 = \frac{1}{T_1} = 1$
 $\omega_2 = \frac{1}{T_2} = 2$
- Phase angel: $\varphi(\omega) = -\arctan(\omega) - \arctan(0.5\omega)$

Bode plot

$$G(p) = \frac{50}{(p+1)(p+2)}$$



References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.

ISBN: 81-7409-107-6

[2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing

[3] Matlab&Simulink: R2015a

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Types of Controlled Systems

Types of controlled systems

We use only first- and second-order differential equation because:

- we are working strictly with linearized systems
- any function can be “broken up” into first-order term or terms with complex roots can be combined together to form a second-order term
- with higher-order models we can construct approximate reduced-order models based on the identification of dominant poles

1. The zero-order system

It has no own dynamic.

Differential equation: $y(t) = k u(t)$

(The output is an intensified input.)

Transfer function: $G(p) = \frac{Y(p)}{U(p)} = k$

2. The first order system

Differential equation: $T \cdot y'(t) + y(t) = k \cdot u(t)$

Where T is time constant and k is steady-state gain

Transfer function: $G(p) = \frac{k}{Tp + 1} = \frac{b_0}{a_1 p + a_0}$

Gain: $K = \frac{b_0}{a_0}$

Time constant: T

Step response: $h(t) = L^{-1} \left\{ \frac{K}{Tp + 1} \frac{1}{p} \right\} = L^{-1} \left\{ \frac{-K}{p + \frac{1}{T}} + \frac{K}{p} \right\} = K \left(1 - e^{-\frac{1}{T}t} \right)$

2. The first order system

Steady state value of step response: $h(t \rightarrow \infty) = \lim_{p \rightarrow 0} p \cdot H(p) = \lim_{p \rightarrow 0} \frac{K}{Tp + 1} = K$

Impulse response: $g(t) = L^{-1}\{G(p)\} = L^{-1}\left\{\frac{K}{Tp + 1}\right\} = L^{-1}\left\{\frac{\frac{K}{T}}{p + \frac{1}{T}}\right\} = \frac{K}{T} e^{-\frac{1}{T}t}$

Steady state value of impulse response: $g(t \rightarrow \infty) = \lim_{p \rightarrow 0} p \cdot G(p) = 0$

Relation between step and impulse response:

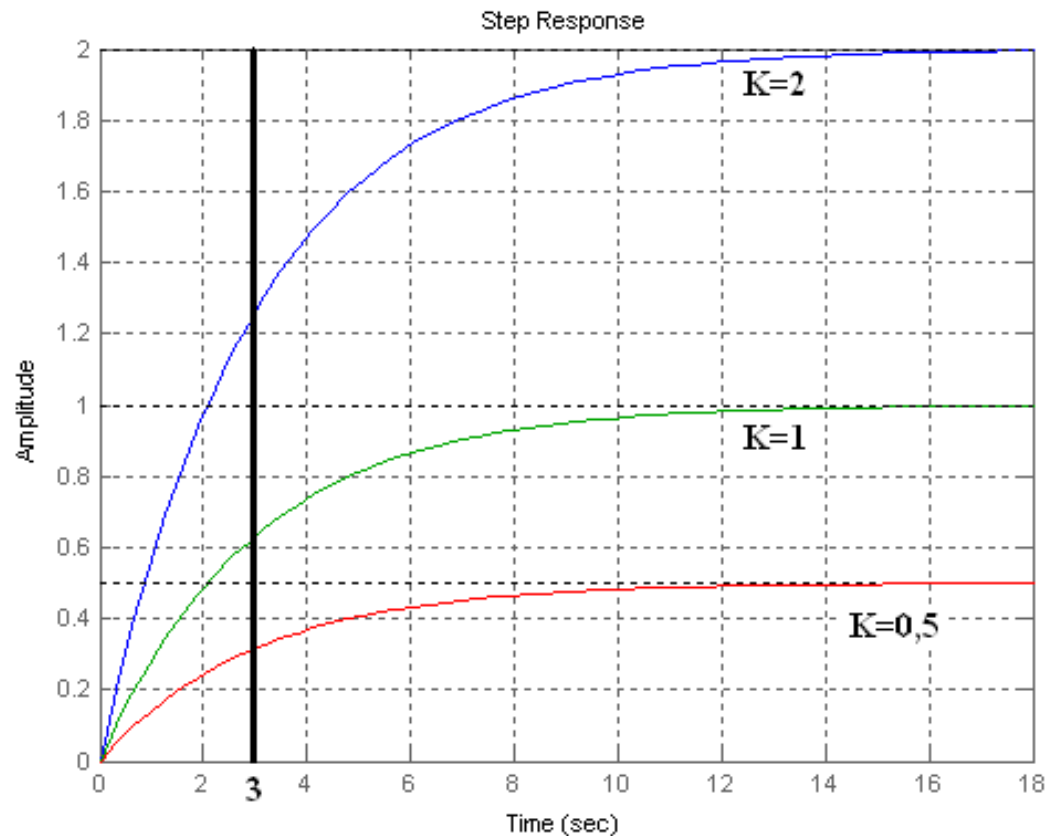
$$g(t) = h'(t) = \left(K - K e^{-\frac{1}{T}t} \right)' = +K e^{-\frac{1}{T}t} \cdot \frac{1}{T}$$

$$h(t) = \int_0^t g(t) dt = \int_0^t \frac{K}{T} e^{-\frac{1}{T}t} dt = \left[\frac{K}{T} (-T) \cdot e^{-\frac{1}{T}t} \right]_0^t = -K e^{-\frac{1}{T}t} + K$$

Step response of a first order transfer function

$$G(p) = \frac{K}{Tp + 1}$$

The effect of changing the steady-state gain K. All curves have T=3.

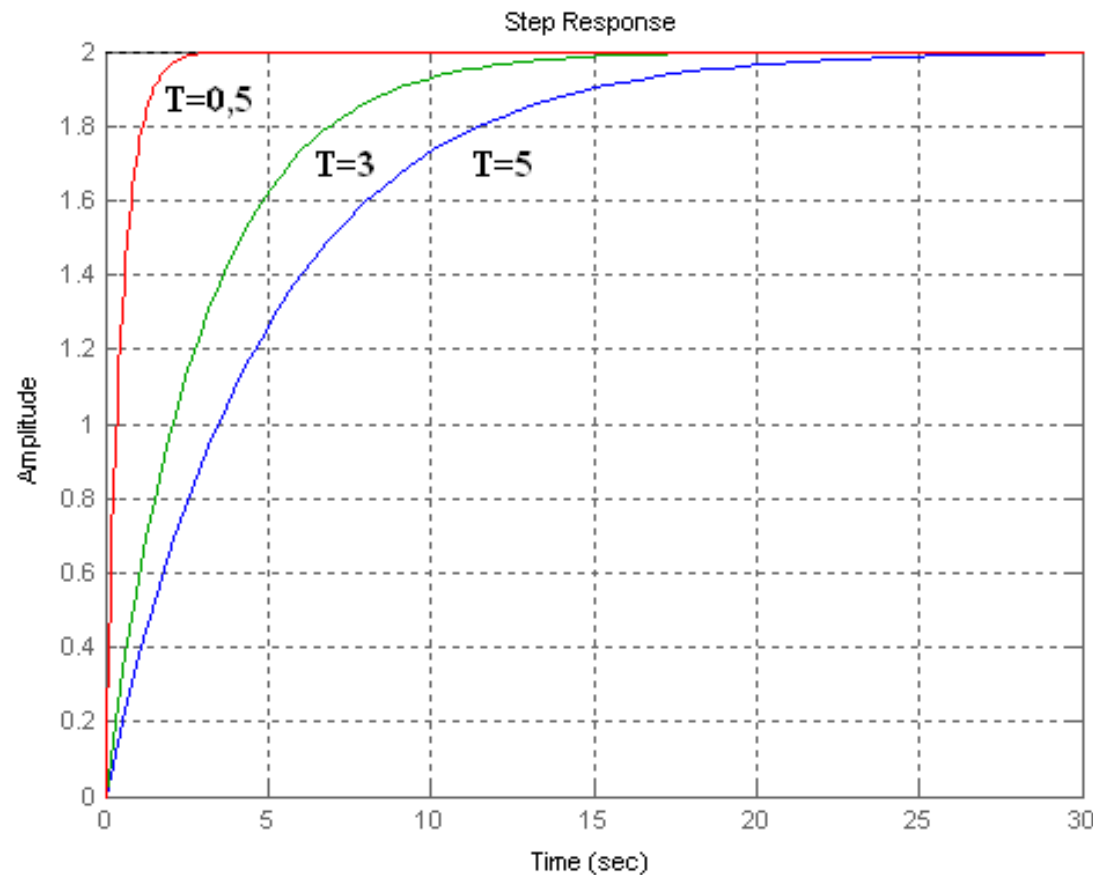


K is changing. At the time when $t=T$ (time constant), the step response reaches 63% of the steady-state value.

Step response of a first order transfer function

$$G(p) = \frac{K}{Tp + 1}$$

The effect of changing the time constant T. All curves have K=2.



System with large time constant gets settled more slowly than system with small time constant.

- At the time when $t = T$ the step response reaches only 63.2% of the steady-state value K .
- System with a large time constant gets settled more slowly.
- System with a small time constant gets settled more quickly.

Time	T	$2T$	$3T$	$4T$	$5T$
Response	63.2 %	86.5 %	95.0 %	98.2 %	99.3 %

3. The second order system

It represents oscillatory behavior.

Differential equation:

$$a_2 y''(t) + a_1 y'(t) + a_0 y = b_0 u(t)$$

Is usually rearranged to take the forms:

$$T_0^2 y''(t) + 2 \xi T_0 y'(t) + y(t) = K u(t)$$

or

$$y''(t) + 2 \xi \omega_n y'(t) + \omega_n^2 y(t) = K \omega_n^2 u(t)$$

The corresponding transfer function is:

$$G(p) = \frac{K}{T_0^2 p^2 + 2 \xi T_0 p + 1} \quad \text{or} \quad \frac{K \omega_n^2}{p^2 + 2 \xi \omega_n p + \omega_n^2}$$

The second order system

System steady-state gain: $K = \frac{b_0}{a_0}$

Time constant: $T > 0$

Natural period of oscillation: $T_0^2 = \frac{a_2}{a_0} = \frac{T_0^2}{1} = \frac{1}{\omega^2}$

Natural inherent (undamped) frequency: $\omega_n > 0$, $\omega_n = \frac{1}{T_0}$

Damping ratio (damping coefficient) ξ :

$$2\xi T_0 = \frac{a_1}{a_0} = \frac{2\xi \omega_n}{\omega_n^2} = \frac{2\xi}{\omega_n} = 2\xi T_0$$

The effect of different values of ξ on system response:

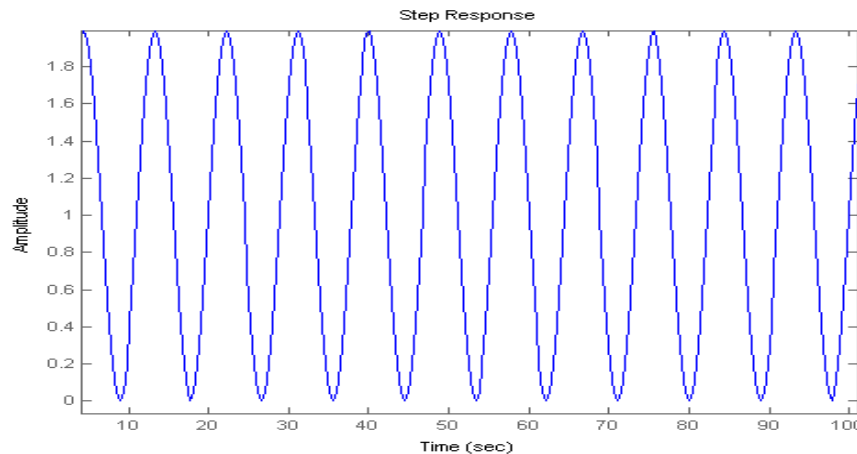
$\xi = 0 \rightarrow$ undamped oscillating system:

Transfer function: $G(p) = \frac{K}{T_0^2 p^2 + 1}$

Two purely imaginary conjugated poles: $p_{1,2} = \sqrt{-\frac{1}{T_0^2}} = \pm \frac{1}{T_0}$

Frequency of oscillation: $\omega_n = 1/T_0$.

Step response:



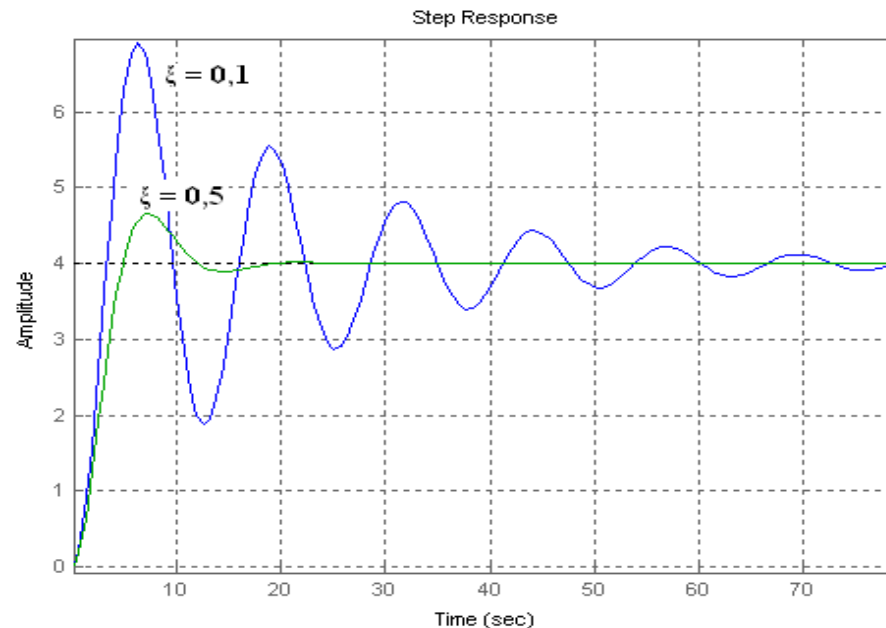
The effect of different values of ξ on system response:

$\xi (0; 1) \rightarrow$ damped oscillating system: two complex conjugated poles

a) For $\xi = 0,1$ and $T_0 = 2$ $\rightarrow G_1(p) = \frac{4}{4p^2 + 2 \cdot 0,1 \cdot 2 p + 1} \Rightarrow p_{1,2} = -0,05 \pm 0,49i$

a) For $\xi = 0,5$ and $T_0 = 2$ $\rightarrow G_2(p) = \frac{4}{4p^2 + 2 \cdot 0,5 \cdot 2 p + 1} \Rightarrow p_{1,2} = -0,25 \pm 0,433i$

Step response:



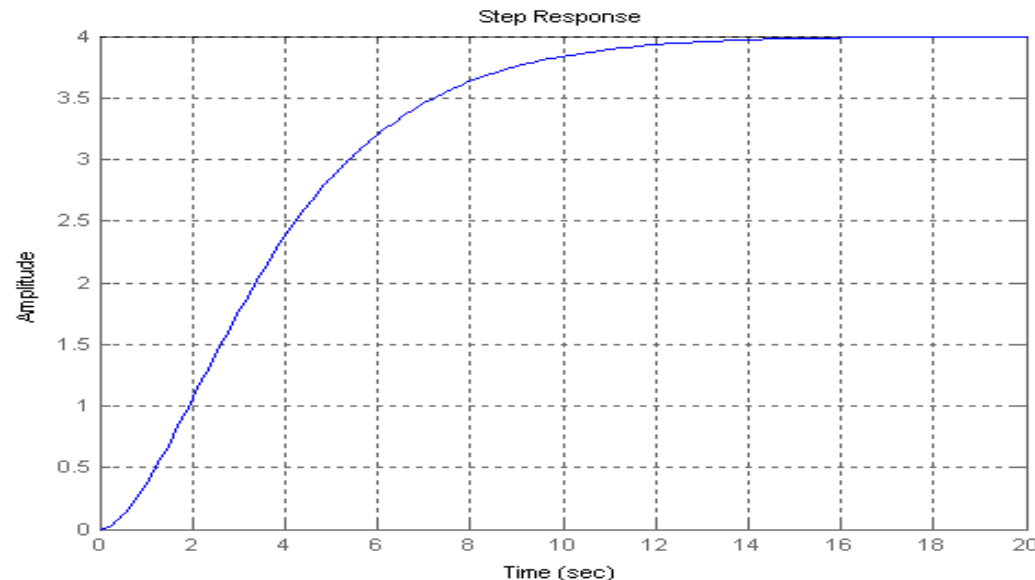
The effect of different values of ξ on system response:

$\xi = 1$ - damped non oscillating system on the limit of periodicity:

Two repeating poles $p_{1,2} = -1/T_0$

Transfer function: $G(p) = \frac{4}{4p^2 + 2 \cdot 1 \cdot 2 p + 1} = \frac{1}{p^2 + p + 1/4} \Rightarrow p_{1,2} = -0,5$

Step response:



The effect of different values of ξ on system response:

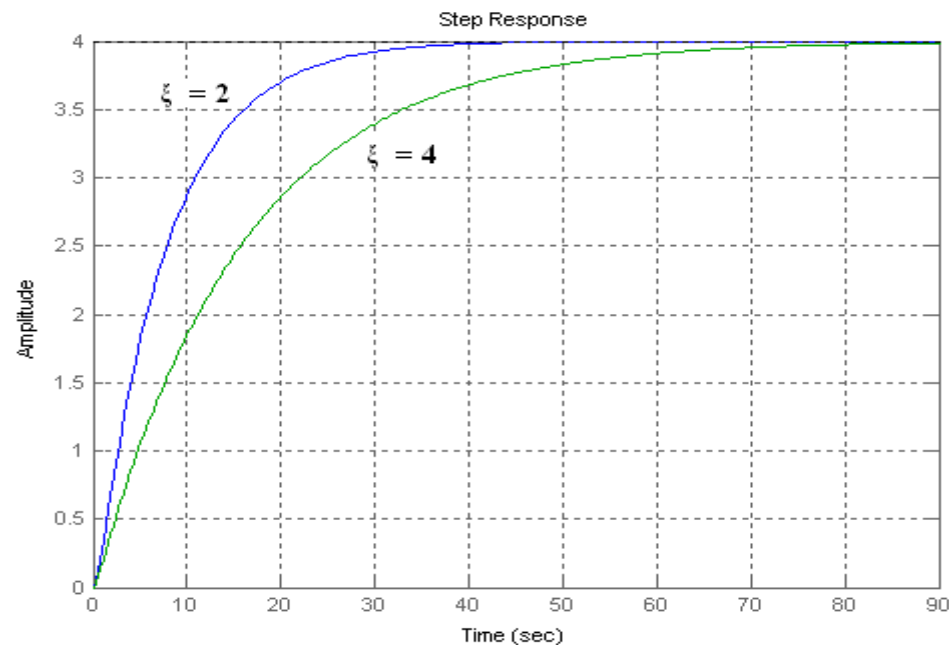
$\xi > 1$ - overdamped non oscillating system:

Transfer function: $G(p) = \frac{K}{4p^2 + 2 \cdot 2 \cdot 2 p + 1} \Rightarrow$

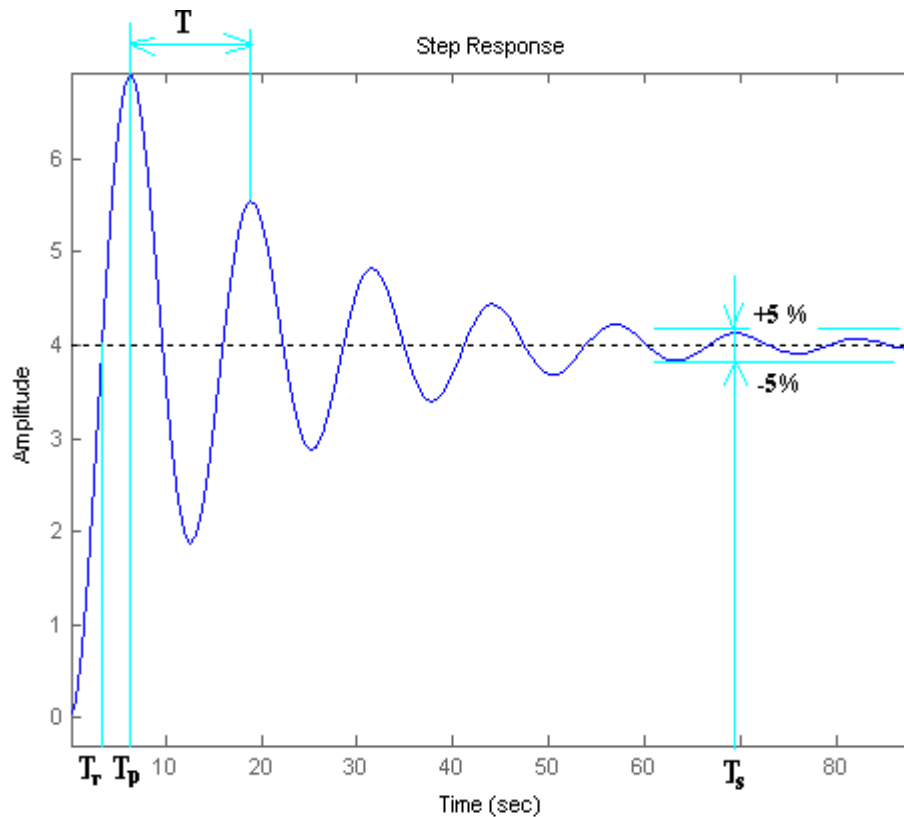
$$p_1 = -1,86$$
$$p_2 = -0,13$$

Two distinct real poles.

Step response:



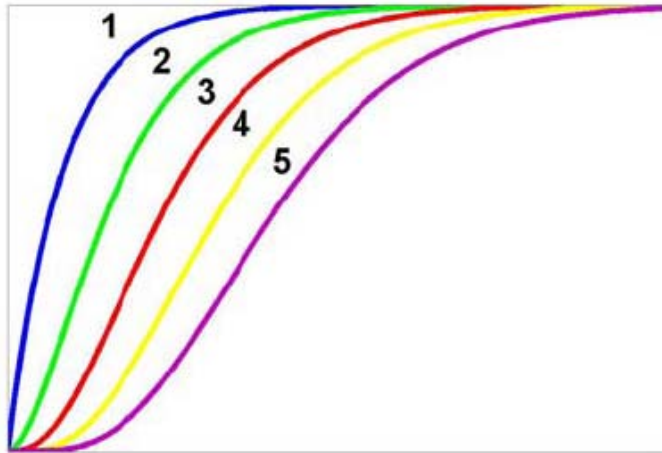
Time-domain features of damped oscillating step response



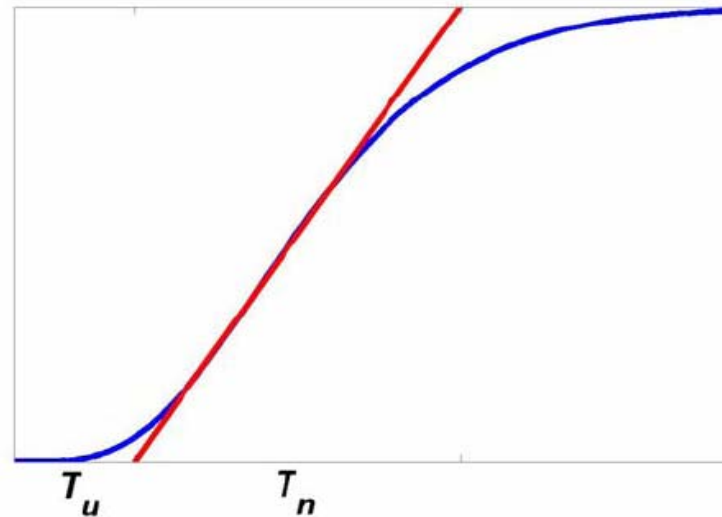
- a) Overshoot
- b) T_p – peak time
- c) T_r – rise time
- d) T – period of oscillation
- e) T_s - settling time

Higher-order systems

Step responses of higher-order systems: $n = 1, 2, 3, 4, 5$



Typical response of higher order-system:



- **T_u – dead time**, the intersection of the tangent line of a step response at the flex point and the time axis
- **T_n – rise time**, the intersection of the tangent line of a step response at the flex point and the steady state value

Higher-order system's approximation

Approximation of non-oscillating systems :

1. Time constant are similar \rightarrow serial connection of \underline{n} identical first-order systems with the same time constant: $T_1 = T_2 = \dots = T_n$

For n-order system:

$$G(p) = \frac{K}{(Tp+1)^n} \rightarrow K \cdot \frac{1}{(Tp+1)} \cdot \frac{1}{(Tp+1)} \cdots \frac{1}{(Tp+1)}$$

Higher-order system's approximation

Approximation of non-oscillating systems :

1. Time constants with large difference: $T_1 \gg T_2 \gg \dots \gg T_n$

a) First-order approximation with dead-time function:

$$G(p) = \frac{K e^{-T_d p}}{(Tp + 1)} \quad \text{Where: } T = T_{max} \text{ the largest time constant}$$
$$T_d = \sum_{i \neq \text{dominant}} T_i$$

b) Second-order approximation with dead-time function:

$$G(p) = \frac{K e^{-T_d p}}{(T_1 p + 1)(T_2 p + 1)} \quad \text{Where } T_1 = T_{max1}, T_2 = T_{max2}$$
$$T_d = \sum_{i \neq 1,2}^n T_i$$

Problem solved

Find the simplest lower-order approximation of the following transfer function:

$$G(p) = \frac{3}{(0,1p + 1)(0,5p + 1)(p + 1)(3p + 1)}$$

The largest time constant: $T=3$,

The corresponding dominant pole: $p = -1/3$

Transport delay : $T_d = \sum T_i = 0,1 + 0,5 + 1 = 1,$

We may approximate as: $G(p) = \frac{3 e^{-1,6p}}{(3p + 1)}$

Transport delay approximation

A time delay is also called transport delay or dead time or transport lag. To handle the time delay we use so-called **Pade' approximation** which expresses the function as a ratio of two polynomials.

The simplest is the first-order Pade' approximation:
$$e^{-T_d p} = \frac{1 - \frac{T_d}{2} p}{1 + \frac{T_d}{2} p}$$

The second-order Pade' approximation:
$$e^{-T_d \cdot p} = \frac{T_d^2 \cdot p^2 - 6T_d p + 12}{T_d^2 \cdot p^2 + 6T_d p + 12}$$

Problem solved

Use the first order Pade' app. To plot the unit-step response for the first order system with dead-time function.

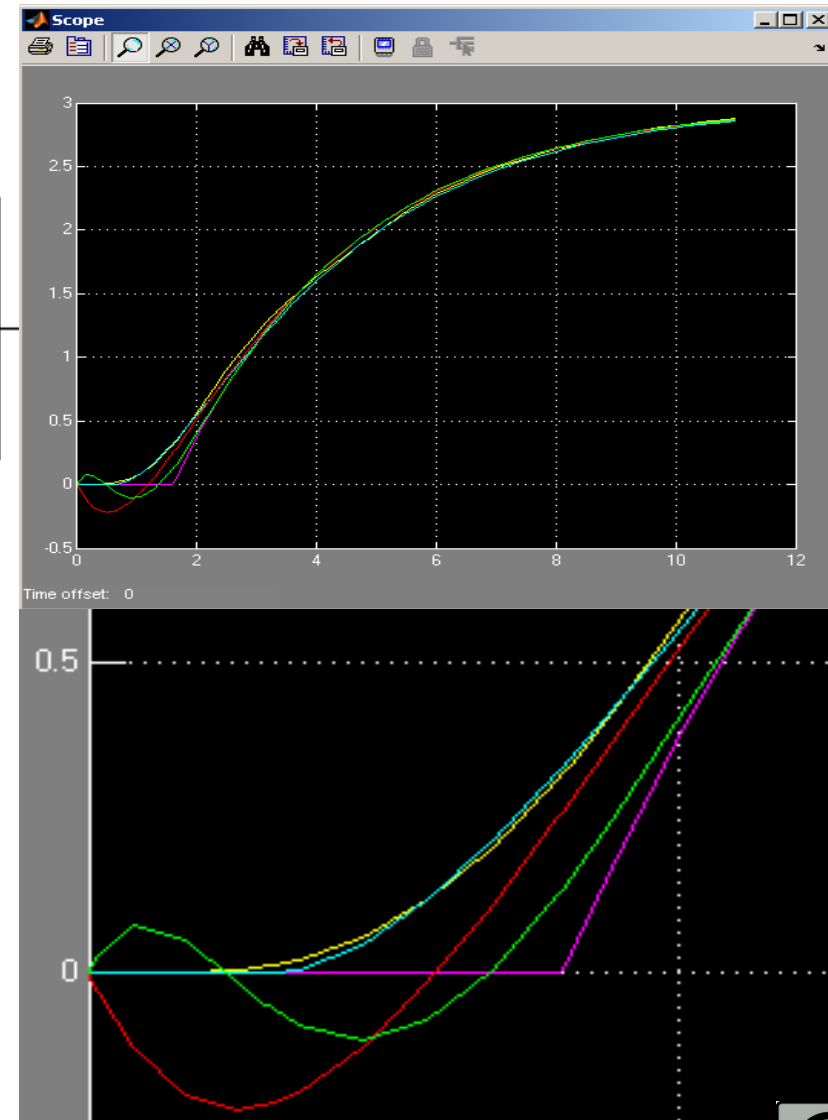
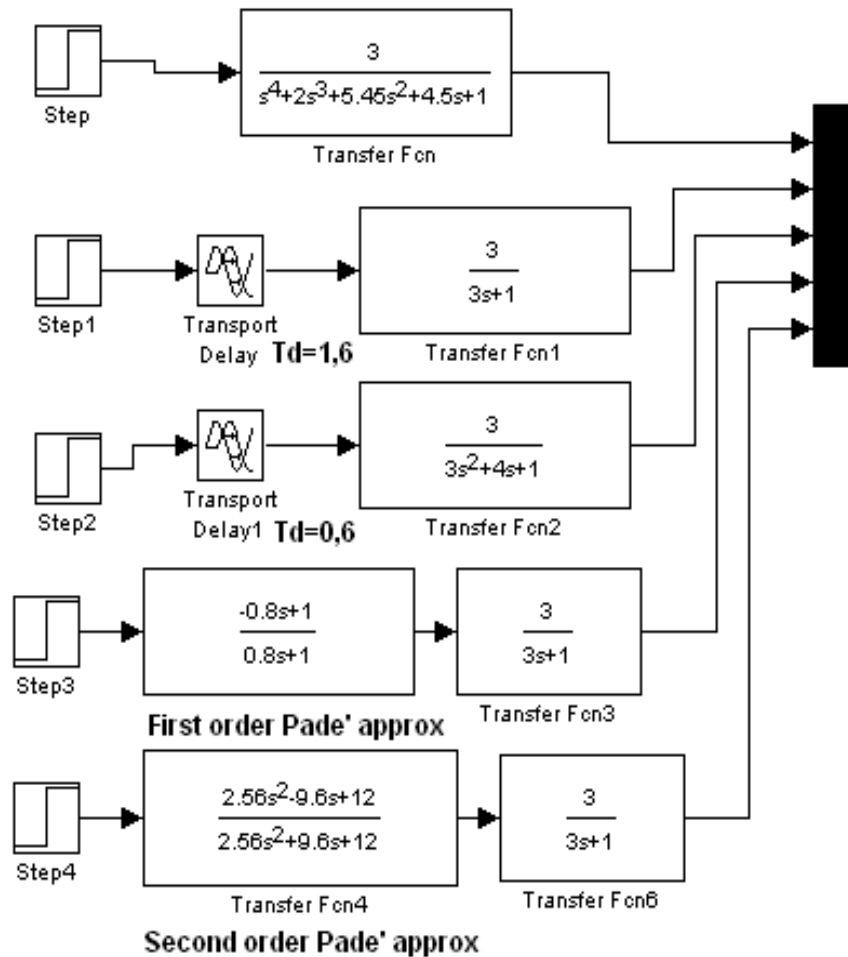
$$G(p) = \frac{e^{-3p}}{10p+1}$$

Solution:

$$e^{-T_d p} = \frac{1 - \frac{T_d}{2} p}{1 + \frac{T_d}{2} p} = \frac{1 - \frac{3}{2} p}{1 + \frac{3}{2} p} = \frac{-1,5p+1}{1,5p+1}$$

$$G(p) = \frac{1}{10p+1} \frac{-1,5p+1}{1,5p+1}$$

Comparison: Time delay approximation and Pade' approximation



References

- [1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.
ISBN: 81-7409-107-6
- [2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing
- [3] Matlab&Simulink: R2015a

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State-space Representation of Controlled Dynamic Systems (Internal Description)

Introduction

Transfer function describes only the relationship between external variables:

- Input variables $U(p)$
- Output variables $Y(p)$

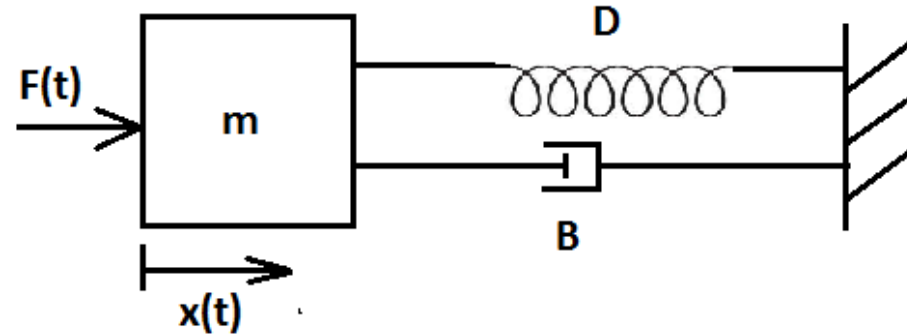
In this approach of analysis initial conditions are considered zero.

State-space representation describes systems using input, output and, in addition, internal (state) variables. The analysis of a control system using state-space approach carried out:

- 1) In time domain considered non-zero initial conditions.
- 2) The high-order differential equation is now written as a set of first differential equations by selecting suitable state variables wherein first order derivative terms are arranged.
- 3) For multiple-input multiple output systems, for non-linear systems the analysis of control systems is more adequate and more suitable.
- 4) Therefore the state-space description is more general than transfer function description.

State-space representation - examples

1. Mechanical system



Differential Equation: $F(t) = F_m(t) + F_B(t) + F_D(t) = m \cdot \frac{d^2x}{dt^2} + B \cdot \frac{dx}{dt} + D \cdot x = m \cdot \ddot{x}(t) + B \cdot \dot{x}(t) + D \cdot x(t)$

Can be rearranged in the following form: $\frac{d^2x}{dt^2} = -\frac{B}{m} \cdot \frac{dx}{dt} - \frac{D}{m} \cdot x - \frac{1}{m} F(t) \Rightarrow a(t) = -\frac{B}{m} \cdot v(t) - \frac{D}{m} \cdot x(t) - \frac{1}{m} F(t)$

For a second order system we need 2 state variables $x_1(t)$ and $x_2(t)$.

Selection of variables – it is recommended, that they are physically measurable.

As state variables are selected: position: $x_1(t) = x(t)$

velocity: $x_2(t) = v(t)$

State-space representation - examples

The representation given by these equations is known as state space representation of the system.

The dynamic behaviour of the system can be determined by two variables $x_1(t) = x(t)$ and $x_2(t) = v(t)$.

In any instant of time $t > t_0$, the state of the system is given by the values of $x_1(t)$ and $x_2(t)$, therefore $x_1(t)$ and $x_2(t)$ are called state variables.

According to the system's description, the relation between the state variables is defined by this first order differential equations:

$$\dot{x}(t) = v(t)$$

$$\dot{v}(t) = a(t) = \frac{1}{m} (F(t) - B \cdot v(t) - D \cdot x(t))$$

$$y(t) = x(t)$$

The state variable $x(t)$ is simultaneously considered to be the output variable of the system.

State-space representation - examples

The state space model, in other words a set of differential equations, can be written in a standard matrix form:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{D}{m} & -\frac{B}{m} \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \cdot F(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \cdot F(t)$$

A general representation of a set of state equation can be expressed in the following form:

$$\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}(t) + \mathbf{D} \cdot \mathbf{u}(t)$$

Where:

x is the state vector ($n \times 1$), or simply the state; the time derivative is denoted by a dot above.
(overdot)

u is the input, or control vector

y is the output

A is the system (plant) matrix, matrix of dynamics of the system, the eigenvalues of A are the poles of transfer function

B is the matrix of inputs

C is the matrix of outputs

D is the input output coupling matrix, the matrix of the direct influence of the input on the output.
 D is usually zero.

When we discuss single-input single-output models (SISO):

u, y are scalar variables

B is a column vector ($n \times 1$)

C is a row vector ($1 \times n$)

A is a square matrix ($n \times n$)

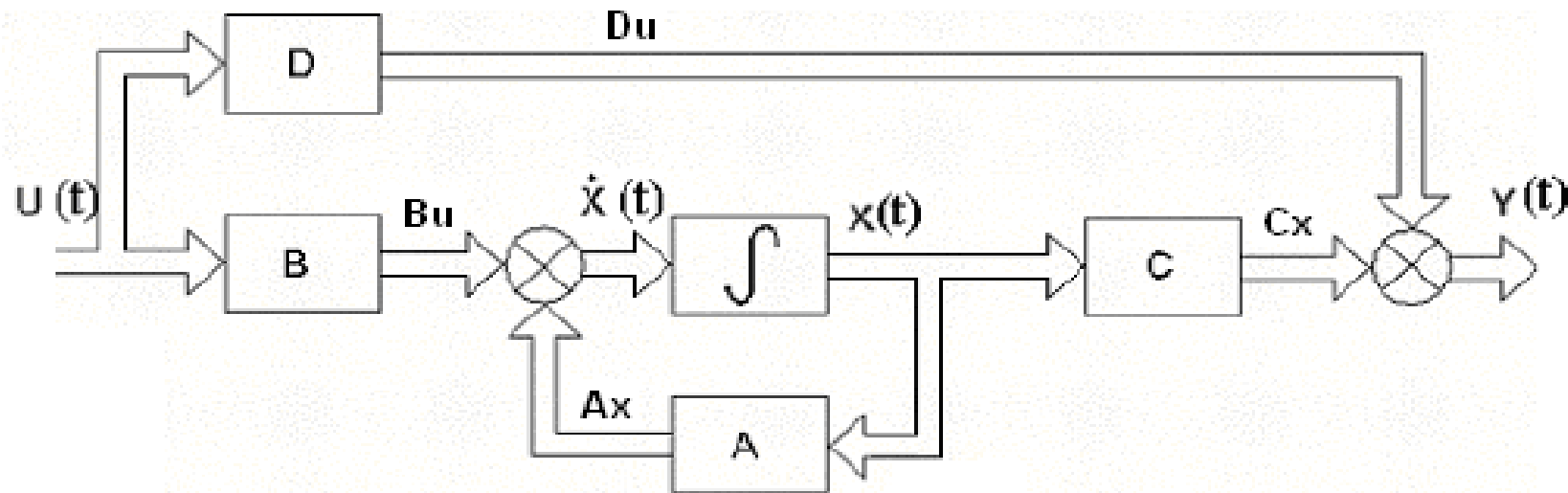
D is a scalar

State-space representation - examples

The block diagram for state equations is shown in the following figure.

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t)$$

$$y(t) = C \cdot x(t) + D \cdot u(t)$$



Transfer function decomposition

- The process of obtaining state model from a transfer function is called **transfer function decomposition**.
- The transfer function decomposition is carried out using different methods. Therefore the corresponding state models of one transfer function can be written in different forms.
- It is demonstrated on the following system, where the transfer function is given in the following forms:

$$G(p) = \frac{Y(p)}{U(p)} = \frac{2p + 1}{p^3 + 9p^2 + 26p + 24}$$

1. **Frobenius' canonical form** of a state space description – Direct decomposition
2. **Jordan's canonical form** of a state space description – Parallel decomposition

Frobenius' canonical form of a state space description

$$G(p) = \frac{Y(p)}{U(p)} = \frac{2p + 1}{p^3 + 9p^2 + 26p + 24}$$

The corresponding differential equation for the transfer function above is of third order; therefore we need three state variables. The state equations are chosen below:

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = x_3(t)$$

$$\dot{x}_3(t) = -24x_1(t) - 26x_2(t) - 9x_3(t) + u(t)$$

And the output is: $y(t) = x_1(t) + 2x_2(t)$

Frobenius' canonical form of state space description

The state model **in matrix form** is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot u(t)$$

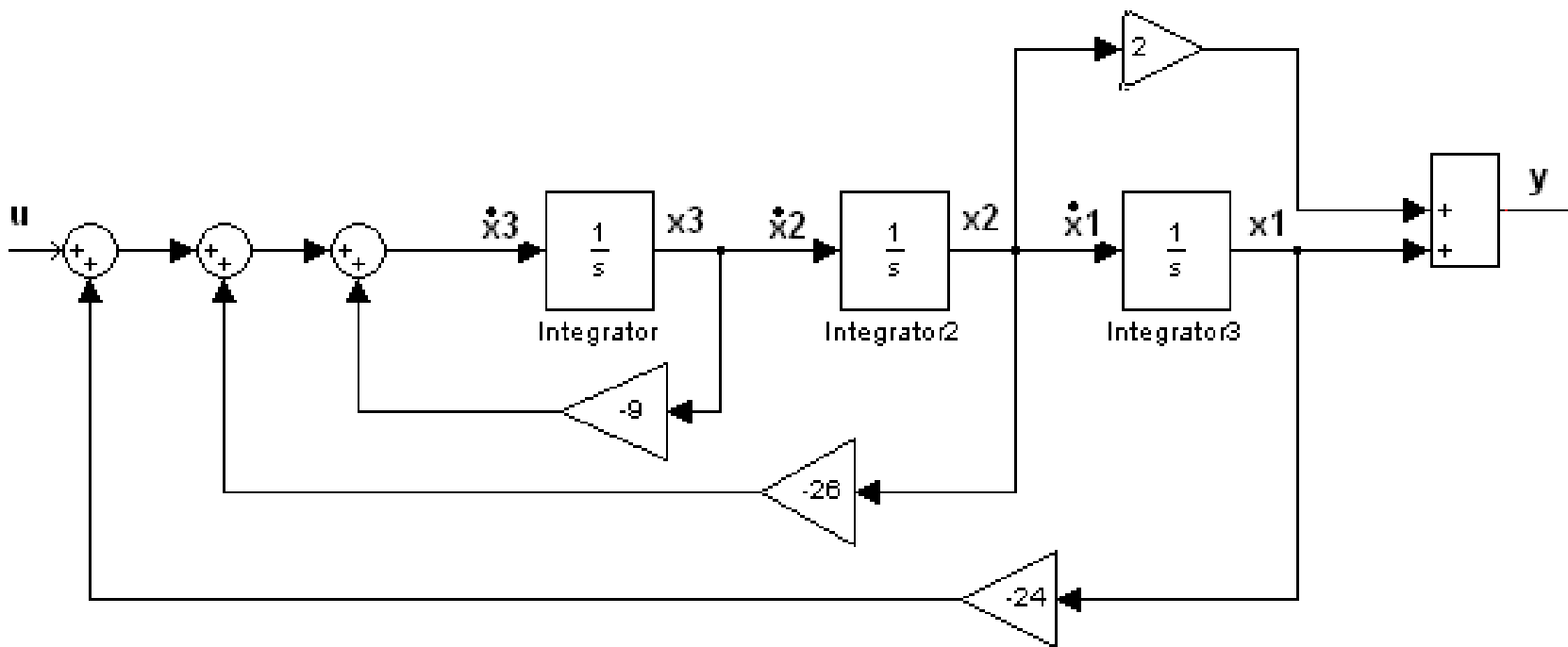
$$y(t) = [1 \quad 2 \quad 0] \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = [1 \quad 2 \quad 0] \quad D = [0]$$

Frobenius' canonical form of state space description

To obtain the block diagram we need three integrators and several summing junctions. The output of each integrator is assigned to a state variable.

Block diagram of Frobenius' form in relation to transfer function: $G(p) = \frac{2p+1}{p^3+9p^2+26p+24}$



Jordan's canonical form of state space description

Jordan's decomposition is carried out by splitting the given transfer function into partial fraction.

$$G(p) = \frac{Y(p)}{U(p)} = \frac{2p+1}{p^3 + 9p^2 + 26p + 24} = \frac{1/2}{p+2} + \frac{-1}{p+3} + \frac{1/2}{p+4}$$

We define the state variables x_1, x_2, x_3 as follows

$$\frac{X_1(p)}{U(p)} = \frac{1}{p+2} \Rightarrow \dot{x}_1(t) + 2x_1(t) = u(t)$$

$$\frac{X_2(p)}{U(p)} = \frac{1}{p+3} \Rightarrow \dot{x}_2(t) + 3x_2(t) = u(t)$$

$$\frac{X_3(p)}{U(p)} = \frac{1}{p+4} \Rightarrow \dot{x}_3(t) + 4x_3(t) = u(t)$$

\Rightarrow

$$\begin{aligned} \dot{x}_1(t) &= -2x_1(t) + u(t) \\ \dot{x}_2(t) &= -3x_2(t) + u(t) \\ \dot{x}_3(t) &= -4x_3(t) + u(t) \end{aligned}$$

Jordan's canonical form of state space description

and output variable $y(t)$ as follows:

$$G(p) = \frac{Y(p)}{U(p)} = \frac{1}{2} \frac{X_1(p)}{U(p)} - \frac{X_2(p)}{U(p)} + \frac{1}{2} \frac{X_3(p)}{U(p)} \Rightarrow Y(p) = \frac{1}{2} X_1(p) - X_2(p) + \frac{1}{2} X_3(p)$$

$$y(t) = \frac{1}{2} x_1(t) - x_2(t) + \frac{1}{2} x_3(t)$$

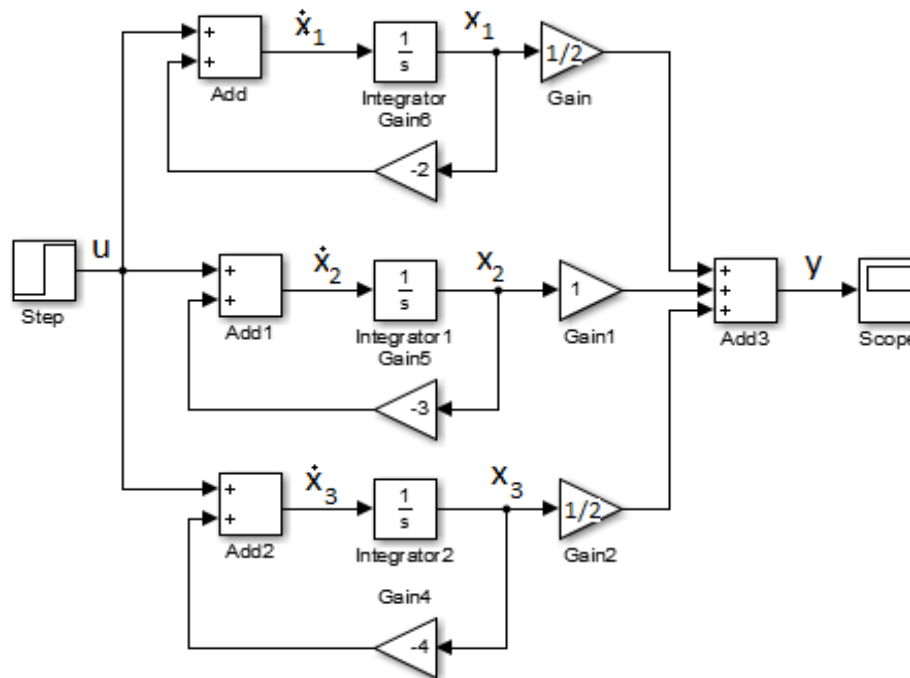
The state model **in matrix form** is:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1/2 & -1 & 1/2 \end{bmatrix}$$
$$y = \begin{bmatrix} 1/2 & -1 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Jordan's canonical form of state space description

Block diagram in Jordan form leads to the parallel connection of the basic elements:



Conclusion

- There is no unique representation of a state space model.
- The characteristic polynomials of both matrixes A are ***identical***.
- The eigenvalues of matrixes A are the poles of the transfer function. They are identical.
- It means, that dynamic behavior in both cases is identical.
- It is a confirmation, that different mathematical techniques provide the same information.

Transfer function decomposition in Matlab.

- ss – state -space description
- zf – transfer function description
- zp – zero-pole description

Matlab conversion:

- $[A,B,C,D] = \text{tf2ss}(\text{num},\text{den})$
- $[A,B,C,D] = \text{zp2ss}(\text{zero},\text{pole})$

- $[\text{num},\text{den}] = \text{ss2tf}(A,B,C,D)$
- $[\text{num},\text{den}] = \text{zp2tf}(\text{zero},\text{pole})$

Problem solved

Derive the state-space representation in **Frobenius** form for the system given by the transfer function.

$$G(p) = \frac{4p^2 + 3p + 1}{2p^2 + 3p + 5}$$

Solution for $m = n$:

1. Order of numerator decreasing: $G(p) = \frac{4p^2 + 3p + 1}{2p^2 + 3p + 5} = 2 + \frac{-3p - 9}{2p^2 + 3p + 5} = 2 + \frac{-3/2p - 9/2}{p^2 + 3/2p + 5/2}$

2. State equations:

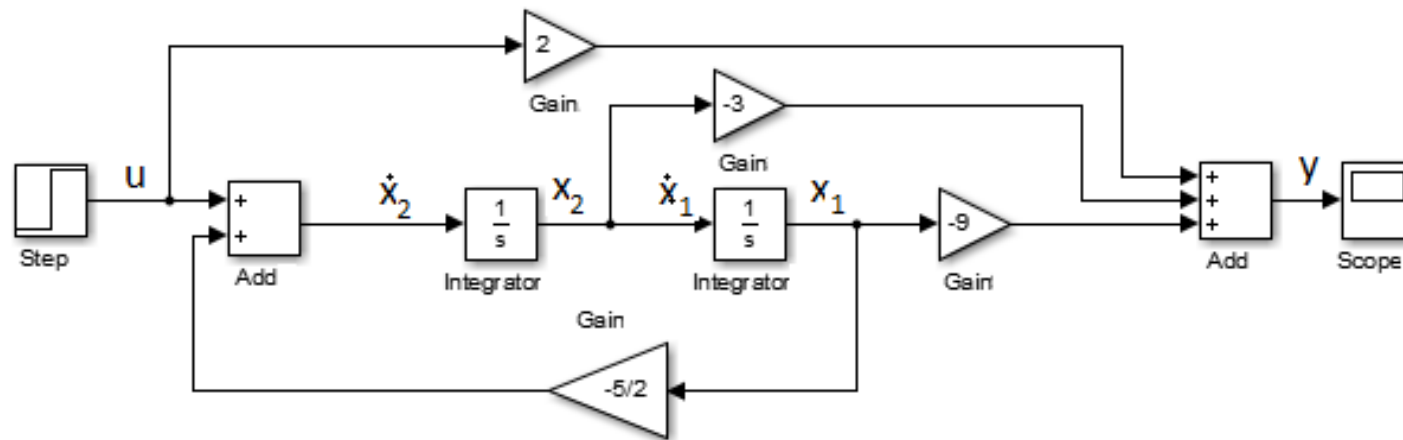
$$\begin{aligned}\dot{x}_1(t) &= x_2 \\ \dot{x}_2(t) &= -\frac{5}{2}x_1 - \frac{3}{2}x_2 + u \\ y(t) &= -9x_1 - 3x_2 + 2u\end{aligned}$$

3. State matrixes:

$$A = \begin{bmatrix} 0 & 1 \\ -5/2 & -3/2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [-9/2 \quad -3/2] \quad D = [2]$$

Problem solved

4. State diagram:



5. Matrix $D=[2]$ represents the direct connection from the input to the output.

Problems

1. Obtain state equations, state matrixes and state diagram in Frobenius form for differential equation.

$$\frac{d^2 y}{dt^2} + \frac{3 dy}{dt} + 4y = \frac{du}{dt} + 3u$$

2. Obtain the state model in Jordan's form using parallel decomposition method.

$$G(p) = \frac{Y(p)}{U(p)} = \frac{p+3}{(p+1) \cdot (p+2)}$$

Relation of state space model to transfer function.

The state equation of a system are given below:

$$\dot{x}(t) = Ax + Bu$$

$$y(t) = Cx + Du$$

Taking Laplace transform of the both sides:

$$p \cdot X(p) - x(0) = A \cdot X(p) + B \cdot u(p)$$

$$Y(p) = C \cdot X(p) + D \cdot U(p)$$

As per definition of transfer function $x(0) = 0$:

$$X(p) = (pI - A)^{-1} \cdot B \cdot U(p) \quad \text{where}$$

$$\Phi(p) = (pI - A)^{-1} \quad \text{is called state - transition matrix}$$

Substituting $X(p)$ in $Y(p)$:

$$Y(p) = [C \cdot (pI - A)^{-1} \cdot B + D] \cdot U(p)$$

where

$$G(p) = [C \cdot (pI - A)^{-1} \cdot B + D] \quad \text{if } D = 0$$

$$G(p) = C \cdot (pI - A)^{-1} \cdot B \quad \text{is transfer function matrix}$$

Problems

1. Derive the state transition matrix and transfer function from the state equations.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2. Derive the state transition matrix from the state equations.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 & 0,2 \\ 0 & 0,2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Revision questions

- Main differences between external and internal description of a system.
- Which forms of internal description do you know?
- Write the state-space description in general form.
- What is x .
- What is A .
- When the matrix $D \neq 0$.
- How can we calculate poles of the system from matrix A .
- How we can calculate transfer function from matrix A, B, C, D .

References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.

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[2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing

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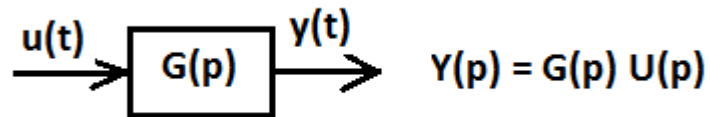


Algebra of Block Diagrams

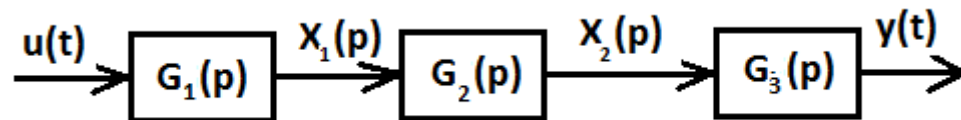
Algebra of block diagrams

We use the block diagram to represent transfer function and lines for unidirectional information transmission.

1. Simple block:



2. Serial connection of systems:



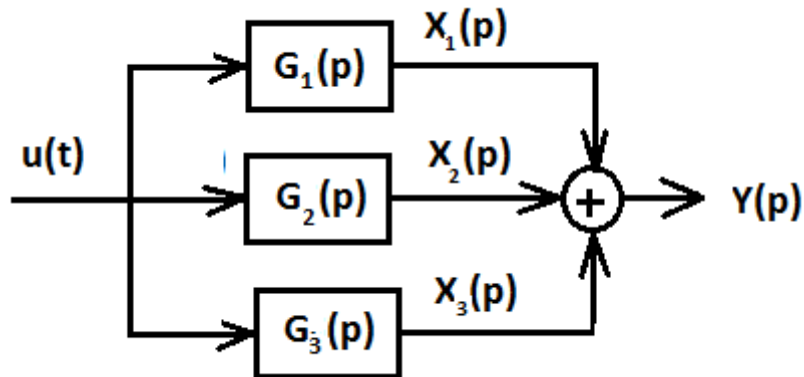
$$Y(p) = G_3(p) X_2(p) = G_3(p) G_2(p) X_1(p) = G_3(p) G_2(p) G_1(p) U(p)$$

$$\text{Total transfer function: } G(p) = G_3(p) G_2(p) G_1(p)$$

Transfer function of the serial connection is multiplying of individual transfer functions.

Algebra of block diagrams

3. Parallel connection of systems:



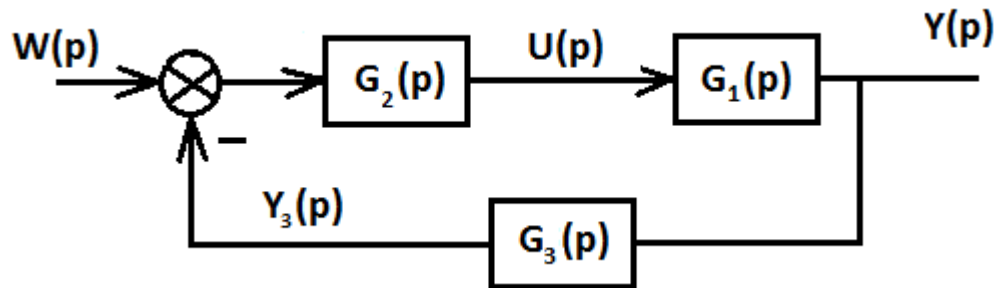
$$Y(p) = X_1(p) + X_2(p) + X_3(p) = G_1(p) U(p) + G_2(p) U(p) + G_3(p) U(p) = (G_1(p) + G_2(p) + G_3(p)) U(p)$$

$$\text{Total transfer function: } G(p) = G_1(p) + G_2(p) + G_3(p)$$

Total transfer function of the parallel connection is the sum of individual transfer function.

Algebra of block diagrams

4. Feedback connection of systems:



$$Y(p) = G_1(p) U(p) = G_1(p) G_2(p) E(p) = G_1(p) G_2(p) (W(p) - Y_3(p)) = \\ G_1(p) G_2(p) W(p) - G_1(p) G_2(p) G_3(p) Y_3(p)$$

$$Y(p) [1 + G_1(p) G_2(p) G_3(p)] = G_1(p) G_2(p) W(p)$$

$$Y(p) = \frac{G_1(p) G_2(p)}{1 + G_1(p) G_2(p) G_3(p)} W(p)$$

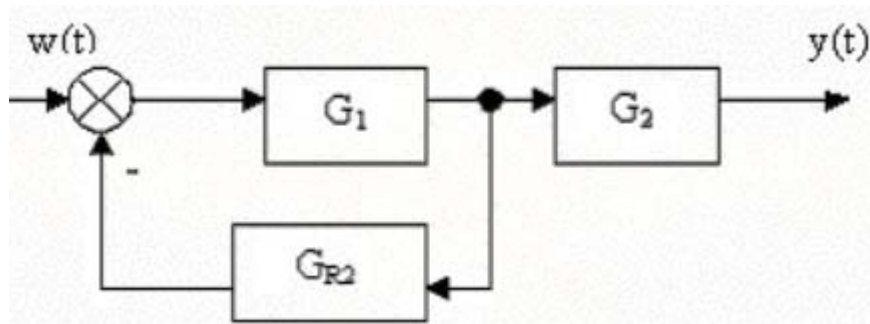
Total transfer function: $G(p) = \frac{G_1(p) G_2(p)}{1 + G_1(p) G_2(p) G_3(p)}$

Mason Rule

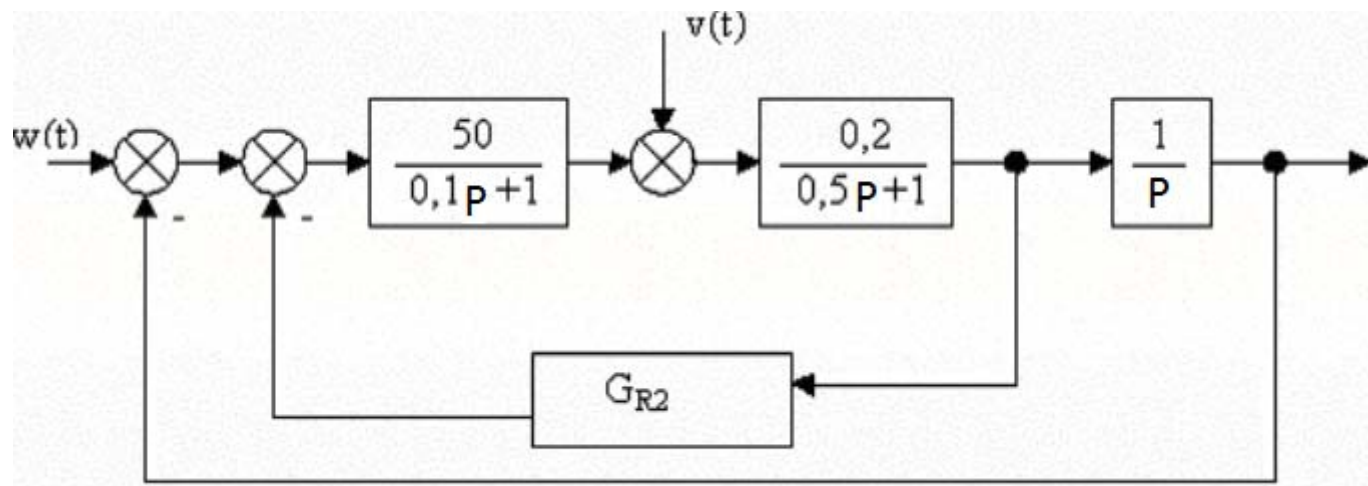
- The important observation is that we close a negative-feedback loop, the numerator consist of the product of all transfer functions along the forward path.
- The denominator is 1 *plus* the product of all the transfer functions in the entire feedback loop (i.e., both forward and feedback paths.) The denominator is also the characteristic polynomial of the closed-loop system.
- If we have positive feedback, the sign in the denominator is *minus*.

Problems

(1)

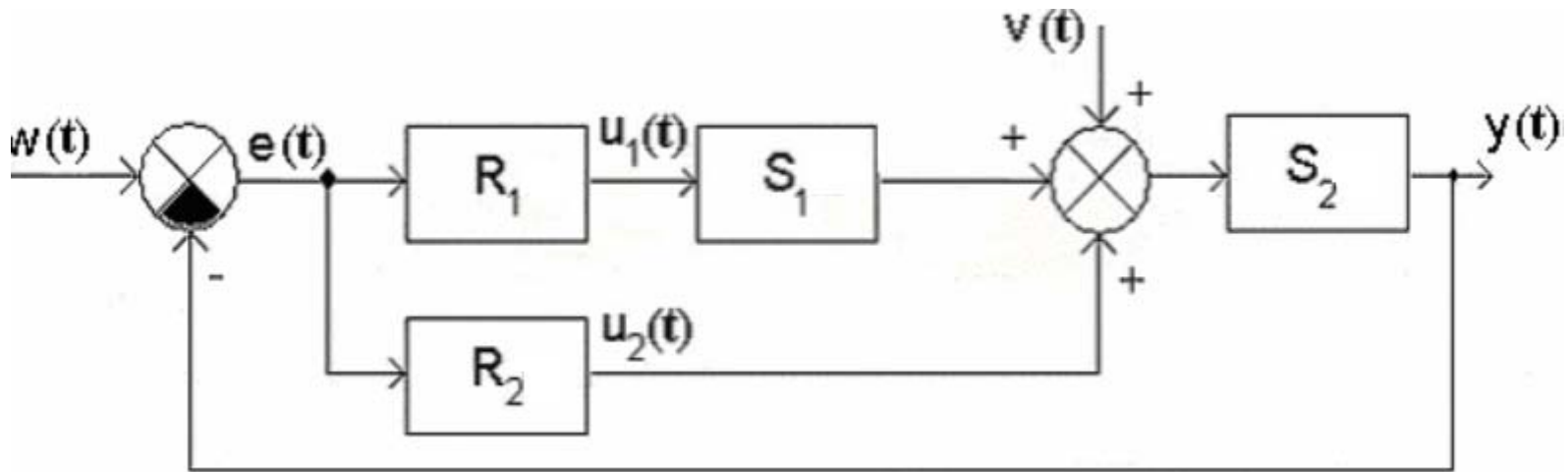


(2)

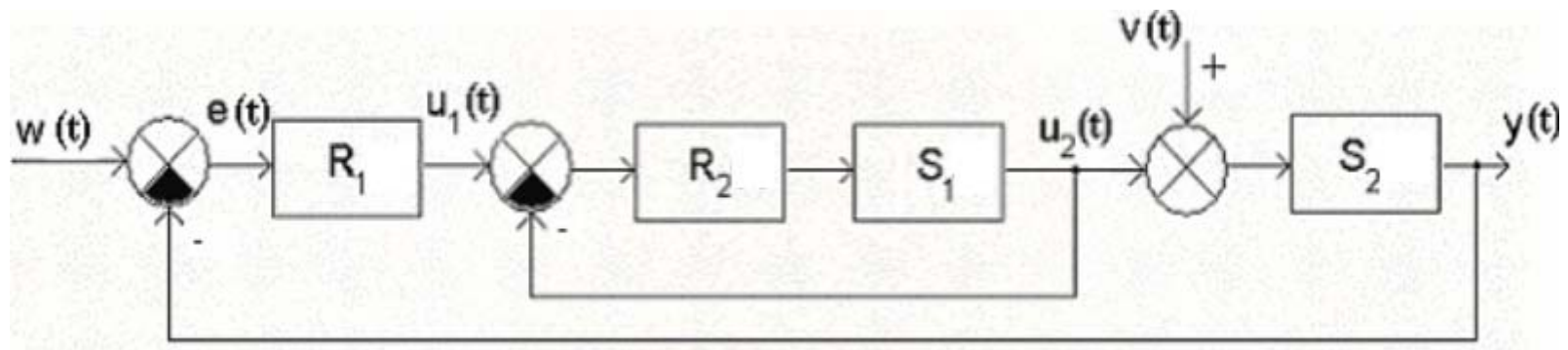


Problems

(3)

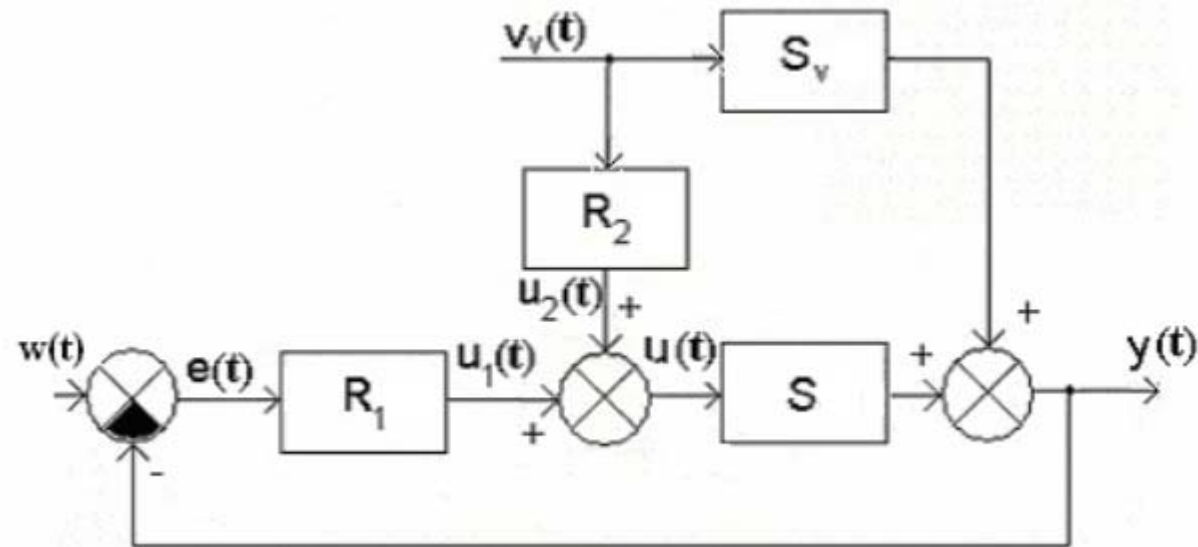


(4)

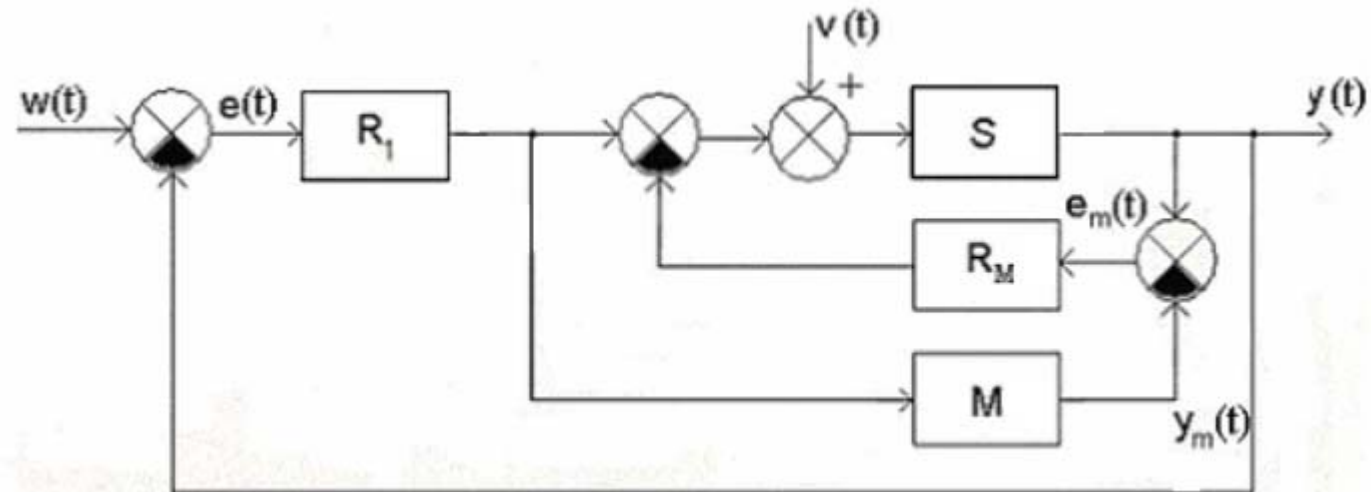


Problems

(5)

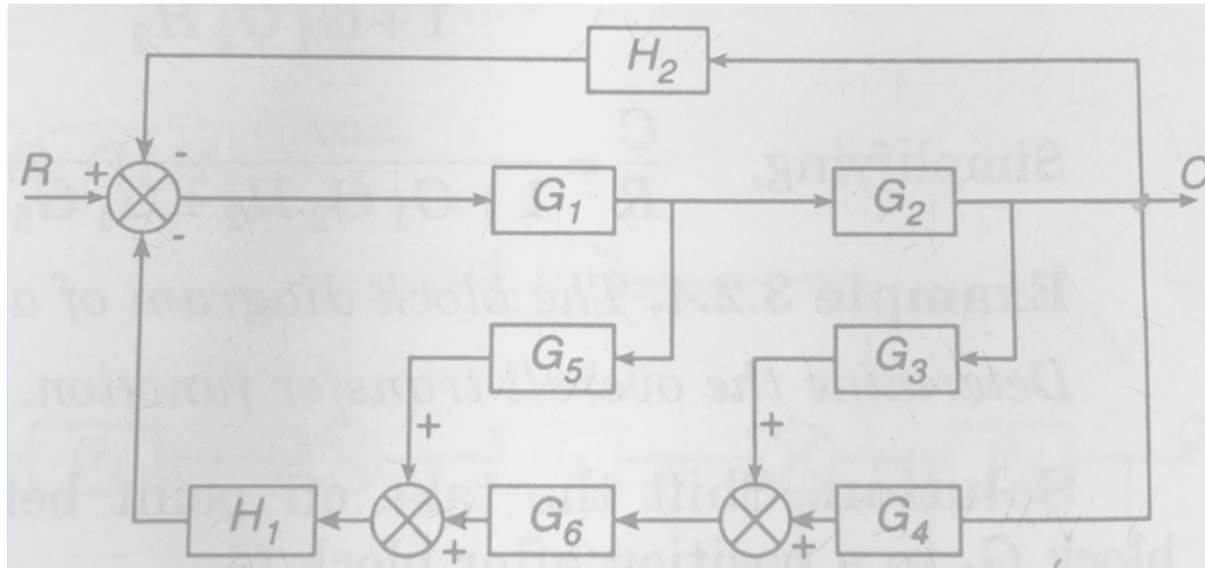


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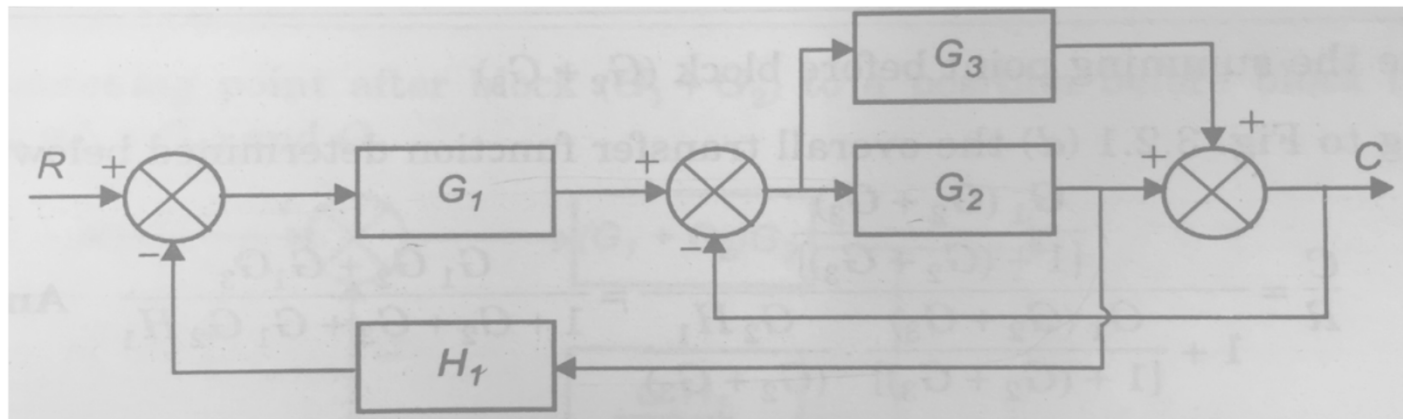


Problems

(7)

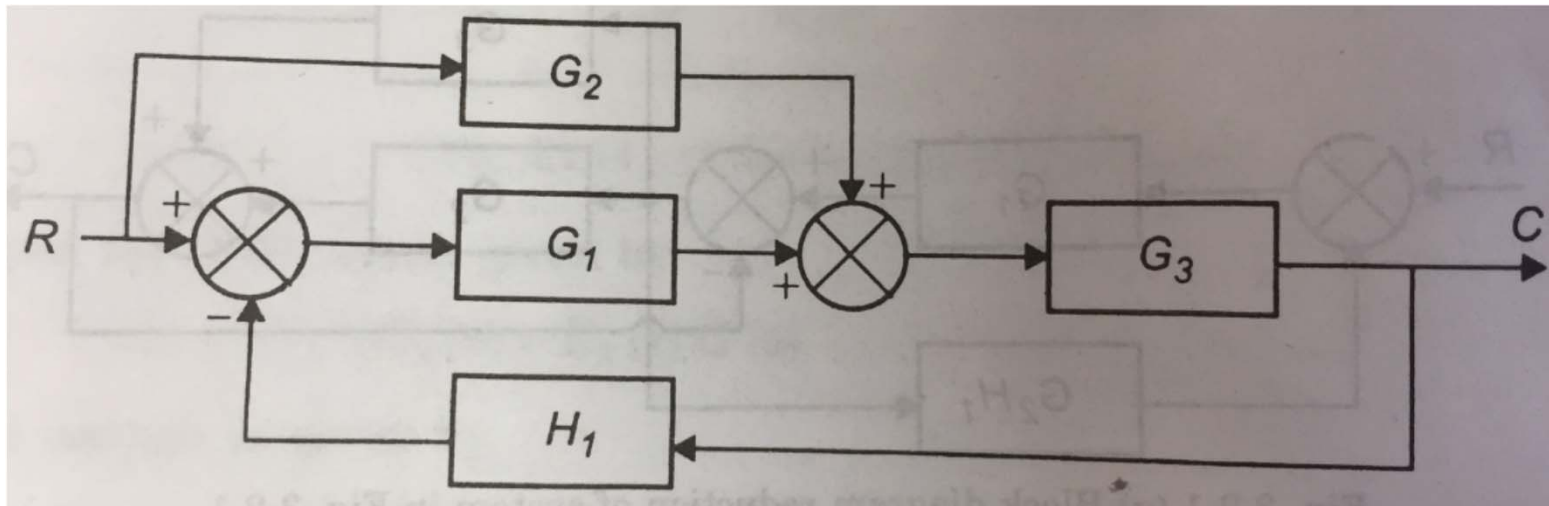


(8)

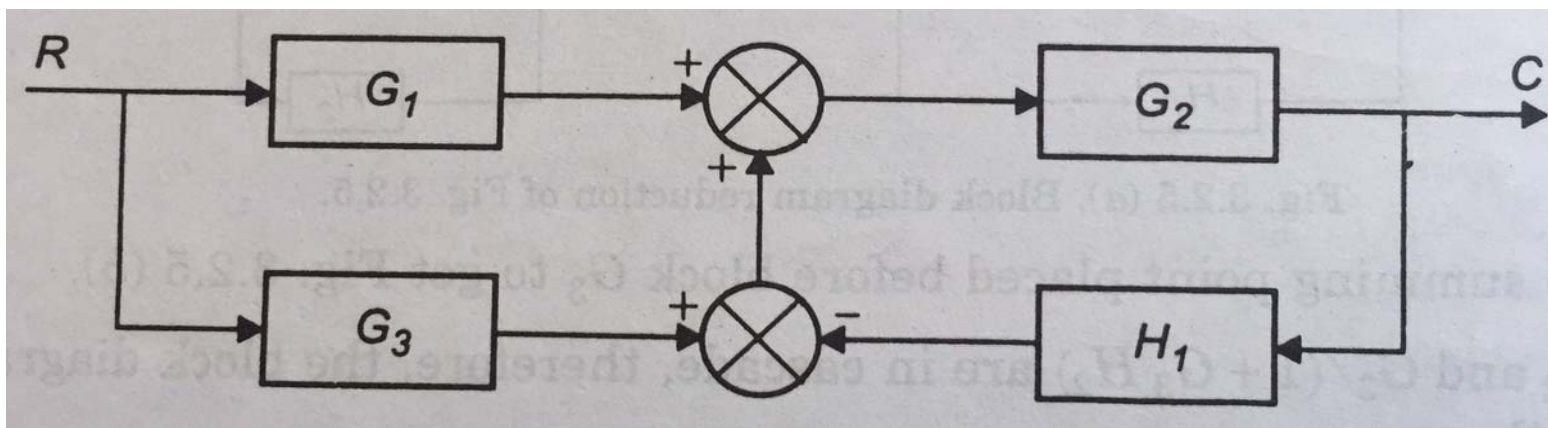


Problems

(9)

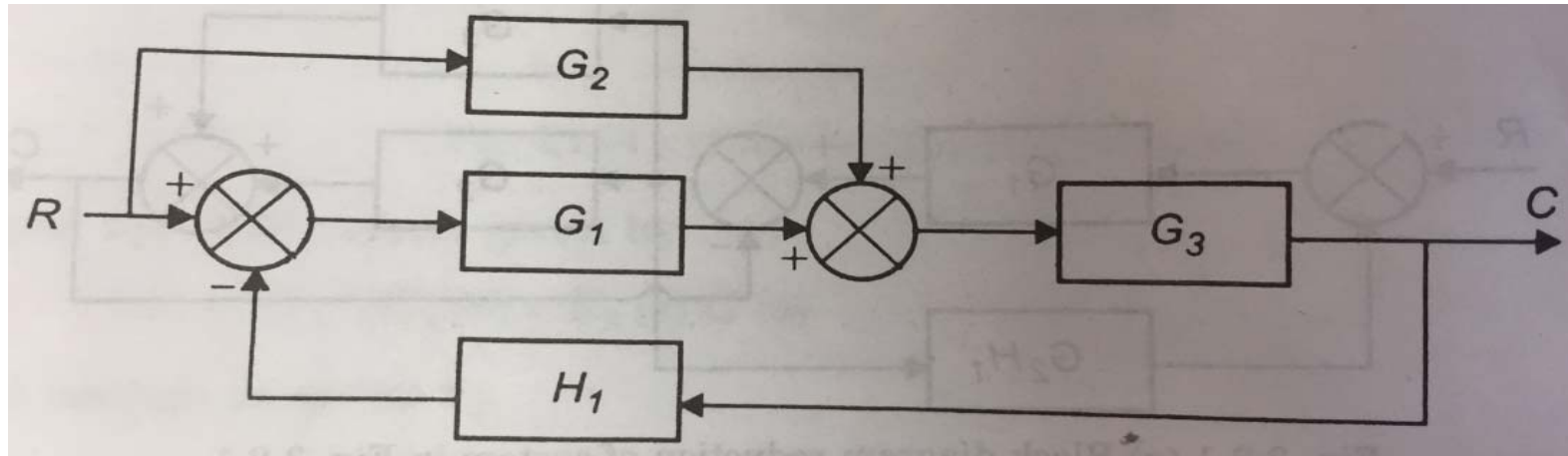


(10)

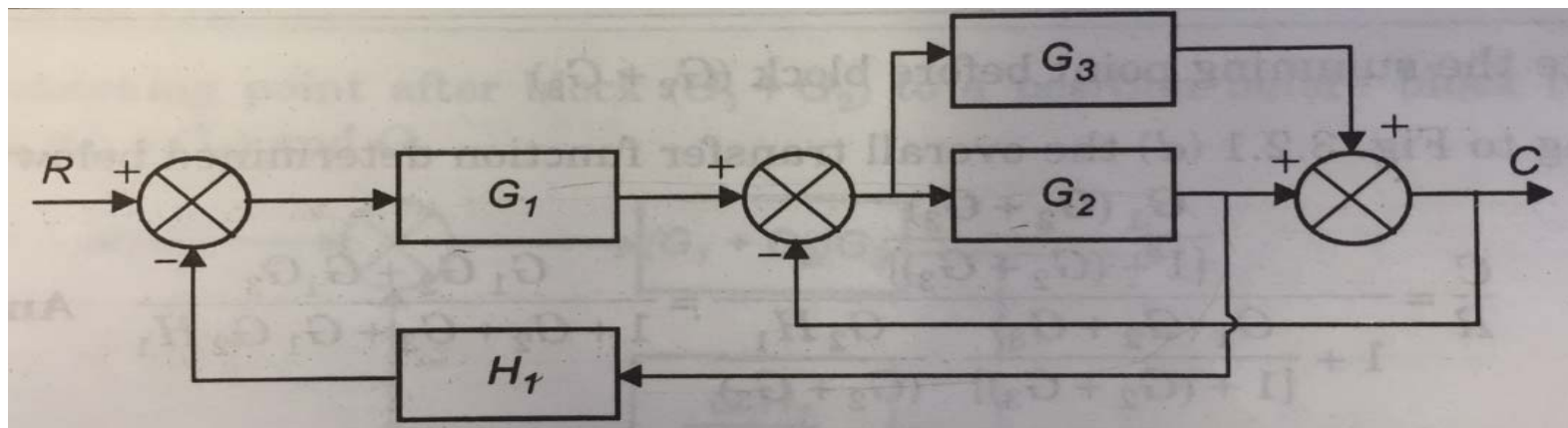


Problems

(11)

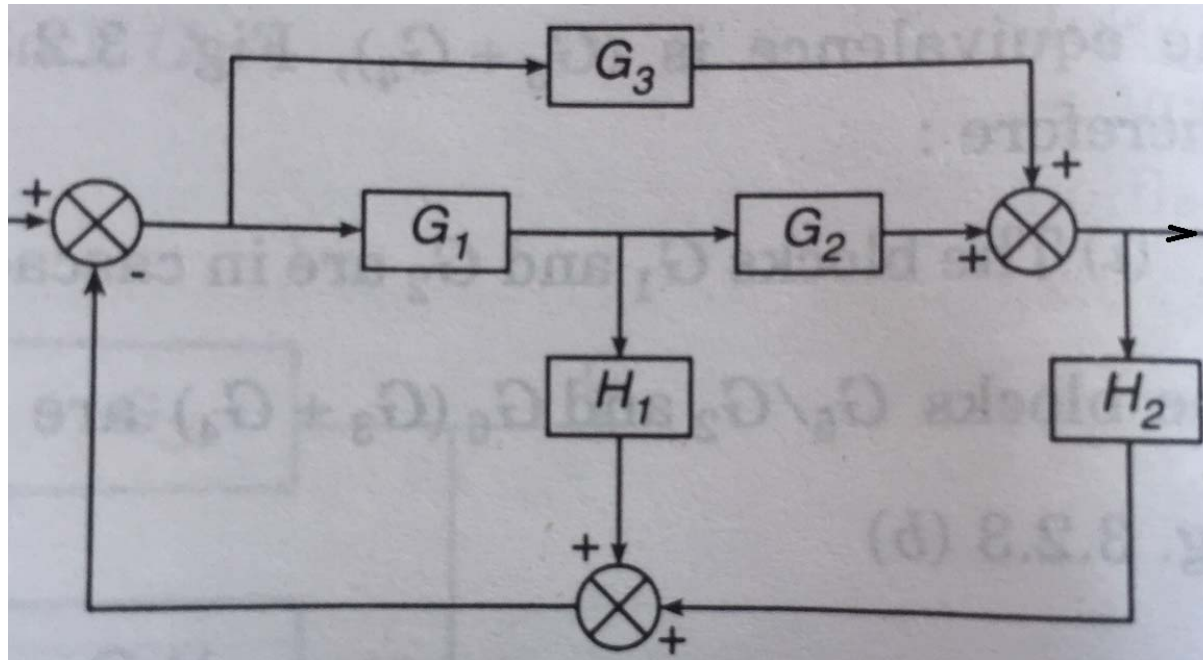


(12)



Problems

(13)



Solutions

$$1) \quad G(p) = \frac{G_1 G_2}{1 + G_1 G_{R1}}$$

$$2) \quad G(p) = \frac{\frac{50}{(0,1p+1)} \cdot \frac{0,2}{(0,5p+1)} \cdot \frac{1}{p}}{1 + \frac{50}{(0,1p+1)} \cdot \frac{0,2}{(0,5p+1)} \cdot \frac{1}{p} + \frac{50}{(0,1p+1)} \cdot \frac{0,2}{(0,5p+1)} \cdot G_{R2}} =$$

$$= \frac{10}{0,05p + 0,6p + p(1 + 10G_{R2}) + 10}$$

$$3) \quad G_W = \frac{R_1 S_1 S_2 + R_2 S_2}{1 + R_1 S_1 S_2 + R_2 S_2} \quad G_V = \frac{S_2}{1 + R_1 S_1 S_2 + R_2 S_2}$$

Solutions

$$4) \quad G_W = \frac{R_1 R_2 S_1 S_2}{1 + R_1 R_2 S_1 S_2 + R_2 S_1}$$

$$5) \quad G_W = \frac{R_1 S}{1 + R_1 S} \quad G_V = \frac{R_2 S + S_V}{1 + R_1 S}$$

$$6) \quad G_W = \frac{R_1 S + R_1 M R_M S}{1 + R_1 S + S R_M + R_1 M R_M S} \quad G_V = \frac{S}{1 + R_1 S + S R_M + R_1 M R_M S}$$

$$7) \quad G = \frac{G_1 G_2}{1 + H_2 G_1 G_2 + G_1 G_5 H_1 + G_1 G_2 G_3 G_6 H_1 + G_1 G_2 G_4 G_6 H_1}$$

$$8) \quad G = \frac{G_1 G_2 + G_1 3}{1 + G_1 G_2 H_1 + G_2 + G_3}$$

Solutions

$$9) \quad G = \frac{G_2 G_3 + G_1 G_2}{1 + G_2 G_3 H_1}$$

$$10) \quad G = \frac{G_2 G_3 + G_1 G_2}{1 + G_2 H_1}$$

$$11) \quad G = \frac{G_2 G_1 + G_3}{1 + G_1 G_2 H_2 + G_1 H_1 + G_3 H_2}$$

References

- [1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.
ISBN: 81-7409-107-6
- [2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing
- [3] Matlab&Simulink: R2015a

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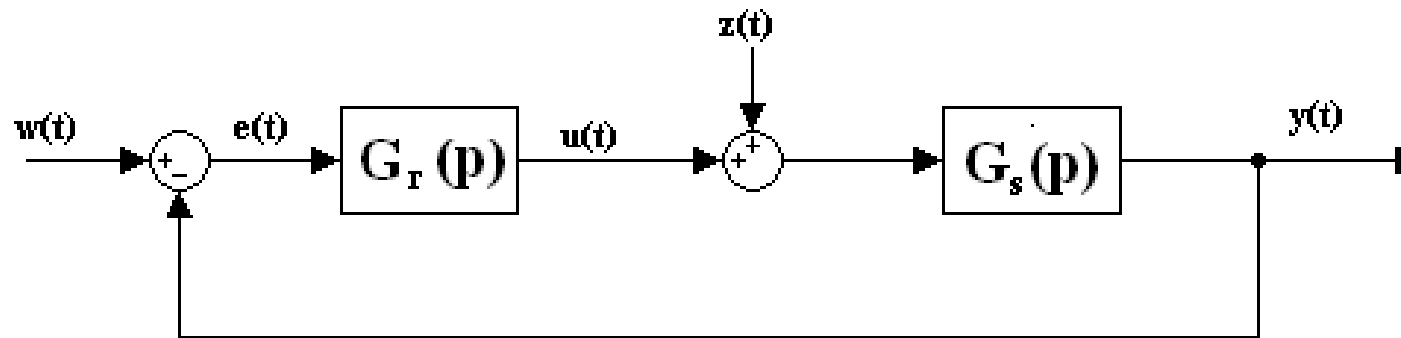


Analysis of Single-loop Control System

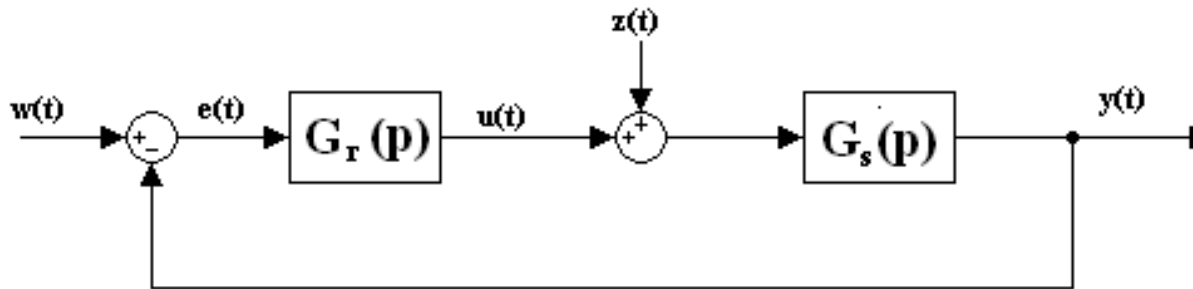
Analysis of Single-loop Control System

Control loop consists of two basic objects:

- Controlled system
 - Controlling system (controller, regulator, actuator)
-
- In a control system the time response is improved by adjusting the feedback parameters.
 - The controlled system and the controlling system are mutually interacted, they created a closed control loop.
 - Any deviation of the output from the reference input is detected by an error detector. The error thus detected is used as actuating signal for control action through a controller. The proportional control action and types of control actions used for improving transient and steady state response of a control system.



- $G_s(p)$ transfer function of the controlled system, system
- $G_r(p)$ transfer function of the controller
- $w(t)$ the system input is the load input (or set point or reference), command variable, our desired steady-state value of the output signal
- $z(t)$ the other system input, disturbance, that can affect our closed-loop response
- $e(t)$ control deviation, error, difference between desired value and controlled value
- $u(t)$ actuating signal
- $y(t)$ controlled value, output of control system



- Control transfer function: G_w

$$G_w(p) = \frac{Y(p)}{W(p)} = \frac{G_r(p)G_s(p)}{1 + G_r(p)G_s(p)} \Rightarrow Y(p) = G_w(p) W(p)$$

- Disturbance transfer function: G_z

$$G_z(p) = \frac{Y(p)}{Z(p)} = \frac{G_s(p)}{1 + G_r(p)G_s(p)} \Rightarrow Y(p) = G_z(p) Z(p)$$

- Deviation transfer function: G_e

$$G_e(p) = \frac{Y(p)}{E(p)} = \frac{1}{1 + G_r(p)G_s(p)} \Rightarrow E(p) = G_e(p) W(p)$$

Problem solved

Determine the control transfer function G_w , error transfer function G_e and disturbance transfer function G_z . Transfer functions of the system and controller are given.

$$G_s(p) = \frac{1}{(p+4)(p+1)} \quad G_R(p) = \frac{3}{(p+3)}$$

$$G_w(p) = \frac{G_r(p)G_s(p)}{1 + G_r(p)G_s(p)} = \frac{\frac{3}{(p+3)} \cdot \frac{1}{(p+4) \cdot (p+1)}}{1 + \frac{3}{(p+3)} \cdot \frac{1}{(p+4) \cdot (p+1)}} = \frac{3}{(p+3) \cdot (p+4) \cdot (p+1) + 3}$$

$$G_e(p) = \frac{1}{1 + G_r(p)G_s(p)} = \frac{1}{1 + \frac{3}{(p+3)} \cdot \frac{1}{(p+4) \cdot (p+1)}} = \frac{(p+1) \cdot (p+4) \cdot (p+1)}{(p+3) \cdot (p+4) \cdot (p+1) + 3}$$

$$G_z(p) = \frac{G_s(p)}{1 + G_r(p)G_s(p)} = \frac{\frac{1}{(p+4) \cdot (p+1)}}{1 + \frac{3}{(p+3)} \cdot \frac{1}{(p+4) \cdot (p+1)}} = \frac{(p+3)}{(p+3) \cdot (p+4) \cdot (p+1) + 3}$$

Problem solved

Determine steady state value of

- a) the output and $G_s(p) = \frac{1}{(p+4)(p+1)}$ $G_R(p) = \frac{3}{(p+3)}$
 b) of the error signal.

Assume the reference input $w(t)=1$ and deviation input $z(t)=1$. Transfer functions of the system and controller are given. Apply final theorem of Laplace transform.

a1)
$$G_w(p) = \frac{Y(p)}{W(p)} = \frac{G_r(p)G_s(p)}{1 + G_r(p)G_s(p)} \Rightarrow Y(p) = G_w(p) W(p)$$

$$y(\infty) = \lim_{p \rightarrow 0} p \cdot Y(p) = \lim_{p \rightarrow 0} p \cdot \frac{3}{(p+3) \cdot (p+4) \cdot (p+1) + 3} \cdot \frac{1}{p} = \frac{1}{5}$$

a2)
$$G_z(p) = \frac{Y(p)}{Z(p)} = \frac{G_s(p)}{1 + G_r(p)G_s(p)} \Rightarrow Y(p) = G_z(p) Z(p)$$

$$y(\infty) = \lim_{p \rightarrow 0} p \cdot Y(p) = \lim_{p \rightarrow 0} p \cdot \frac{p+3}{(p+3) \cdot (p+4) \cdot (p+1) + 3} \cdot \frac{1}{p} = \frac{1}{5}$$

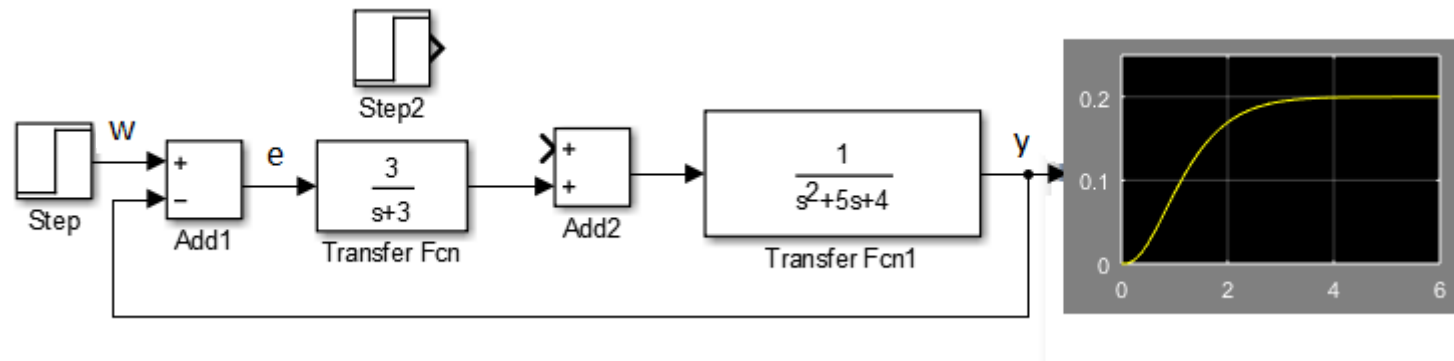
b)
$$G_e(p) = \frac{E(p)}{W(p)} = \frac{1}{1 + G_r(p)G_s(p)} \Rightarrow E(p) = G_e(p) W(p)$$

$$e(\infty) = \lim_{p \rightarrow 0} p \cdot E(p) = \lim_{p \rightarrow 0} p \cdot \frac{(p+3) \cdot (p+4) \cdot (p+1)}{(p+3) \cdot (p+4) \cdot (p+1) + 3} \cdot \frac{1}{p} = \frac{4}{5}$$

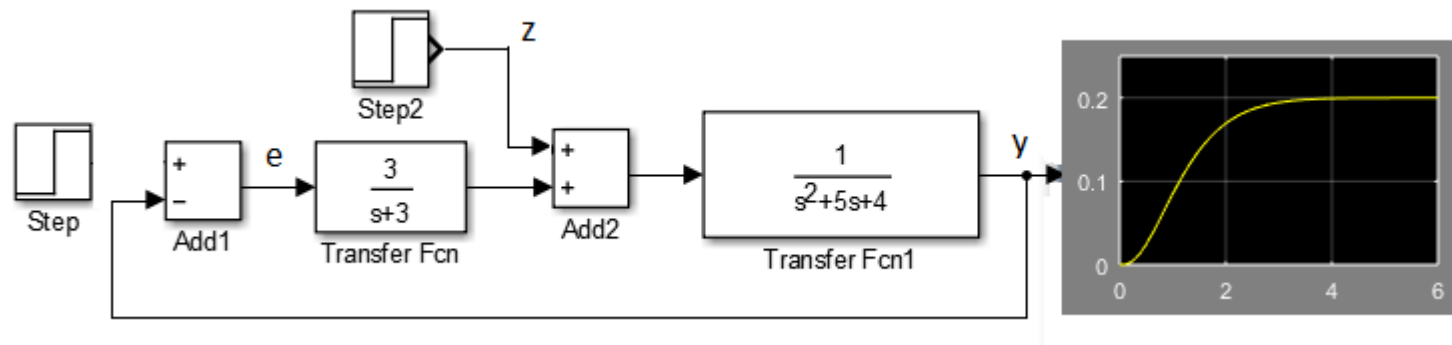
Problem solved

Simulation of step responses in Matlab.

a1) step response – steady state value of output if $w(t)=1$



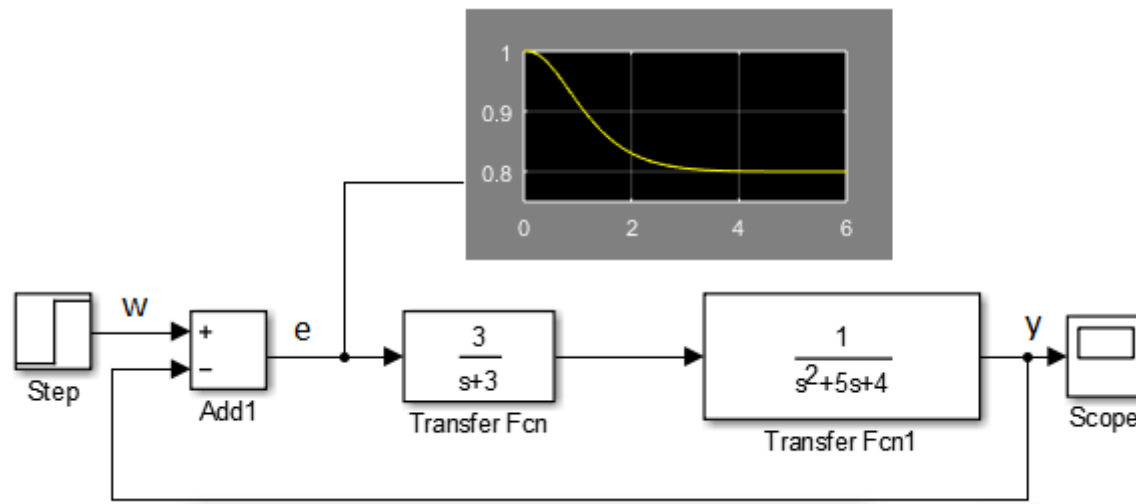
a2) step response – steady state value of output if $z(t)=1$



Problem solved

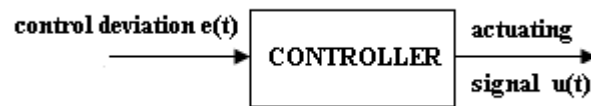
Simulation of the step response in Matlab.

b) Steady state value of error signal, if $w(t)=1$



Linear analogue controllers

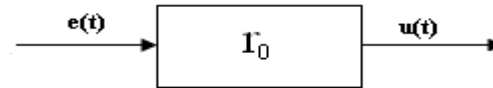
Any deviation of the output from reference input is detected by an error detector. The error is used as an actuating signal for control action through a controller.



There are three simple types of controllers according to the way they process a signal.

- 1) Proportional control:** actuating signal is proportional to the error signal
- 2) Derivative control:** actuating signal is proportional to the derivative of error signal
- 3) Integral control:** actuating signal is proportional to the integral of error signal

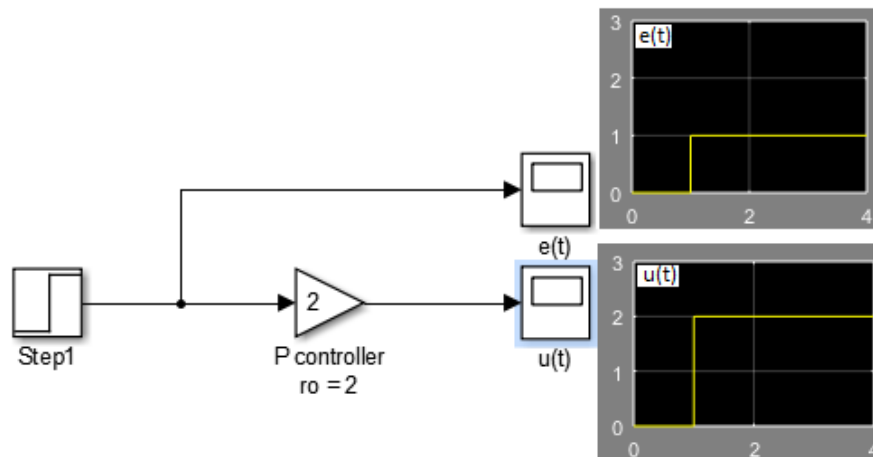
1. Proportional control



Differential equation of ideal P-controller is: $u(t) = r_0 \cdot e(t)$

Transfer function of ideal P-controller is: $U(p) = r_0 E(p) \Rightarrow G_r(p) = \frac{U(p)}{E(p)} = r_0$

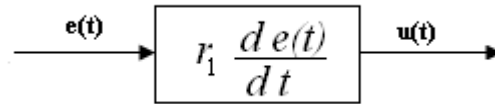
Step response:



General features of P-controller.

- We expect that P-controller will improve or accelerate the response of a process. The larger r_0 is the faster and more sensitive response with respect to $e(t)$ -given error.
- If r_0 is too large we expect the control compensation to overreact, leading to oscillatory response, in the worst case, the system may become unstable.
- There are physical limits (for voltage or current) – the control system is **saturated**.
- We may expect a system with only proportional controller to have **steady-state error** (offset)

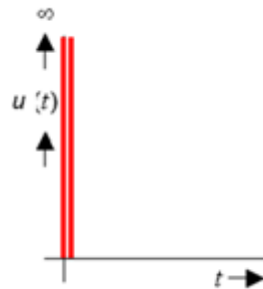
Derivative D-controller.



Differential equation of ideal D-controller is: $u(t) = r_1 \frac{de(t)}{dt}$

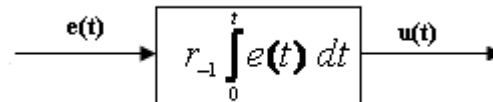
Transfer function of ideal D-controller is: $U(p) = r_1 p E(p) \Rightarrow G_r(p) = \frac{U(p)}{E(p)} = r_1 p$

Step response:
(ideal)



Derivation is never used by itself !

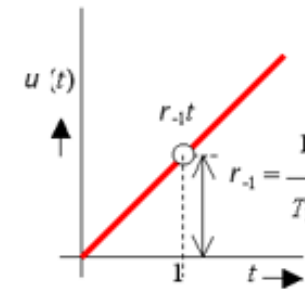
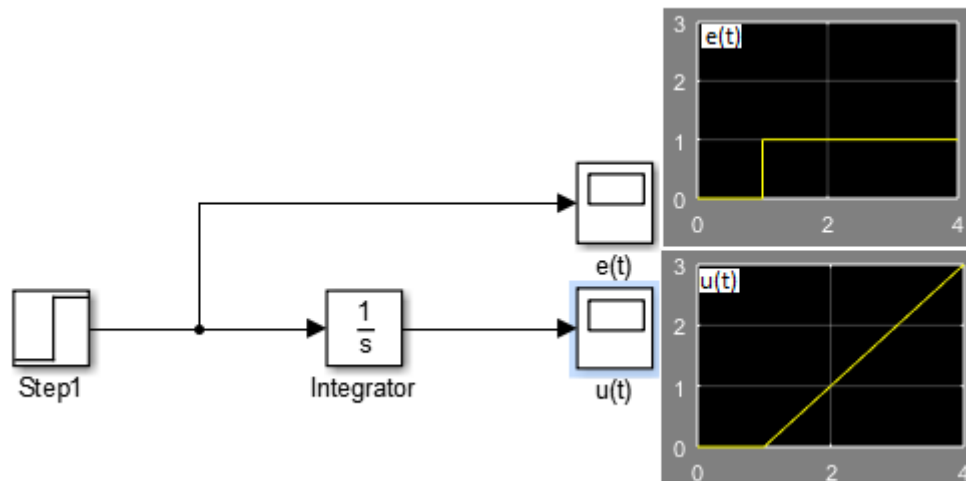
Integral I-controller.



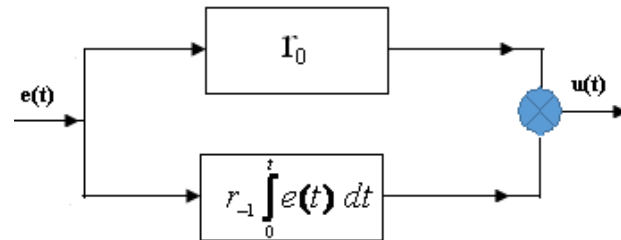
Differential equation of ideal P-controller is: $u(t) = r_{-1} \int_0^t e(t) dt$

Transfer function of ideal P-controller is: $U(p) = r_{-1} \frac{1}{p} E(p) \Rightarrow G_r(p) = \frac{U(p)}{E(p)} = \frac{r_{-1}}{p}$

Step response:



Proportional-Integral: PI-controller

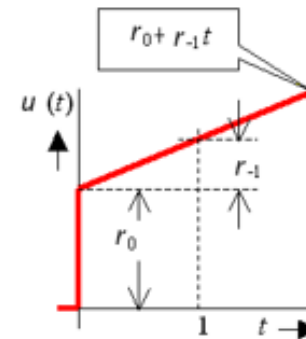
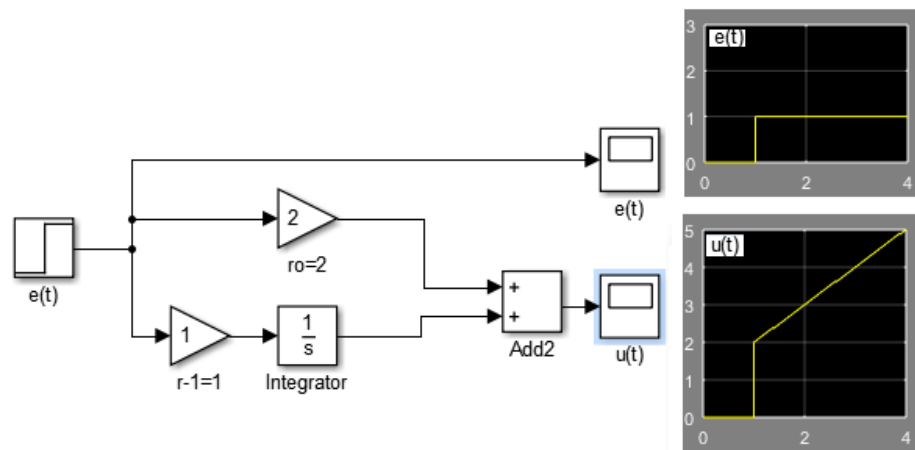


Differential equation of ideal PI-controller is: $u(t) = r_0 e(t) + r_{-1} \int_0^t e(t) dt$

Transfer function of ideal PI-controller is: $U(p) = r_0 E(p) + r_{-1} \frac{1}{p} E(p) \Rightarrow$

$$G_r(p) = \frac{U(p)}{E(p)} = r_0 + \frac{r_{-1}}{p} = \frac{r_0 p + r_{-1}}{p}$$

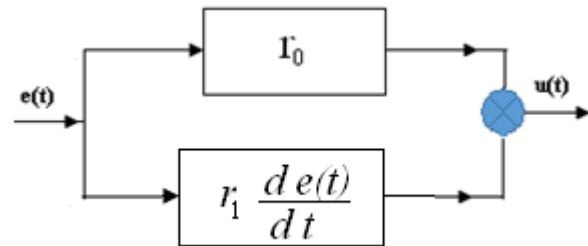
Step response:



General features of PI-controller

- r_{-1} is the integral time constant
- The integral action is such that we accumulate the error from $t=0$ to the present. Thus the integral is not necessarily zero even if the current error is zero.
- PI-controllers can **eliminate offset**.
- The elimination of the offset is usually at the expense of a more underdamped system response. The oscillatory response may have a short rise time, but is penalized by excessive overshoot or exceedingly **long settling time**.

Proportional-Derivative: PD-controller.



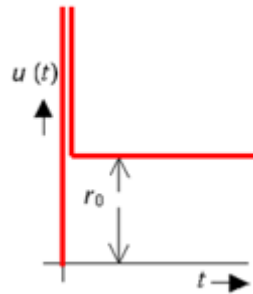
Differential equation of ideal PD-controller is: $u(t) = r_0 e(t) + r_1 \frac{de(t)}{dt}$

Transfer function of ideal PD-controller is:

$$U(p) = r_0 E(p) + r_1 p E(p) \Rightarrow$$

$$G_r(p) = \frac{U(p)}{E(p)} = r_0 + r_1 p$$

Step response:
(ideal)



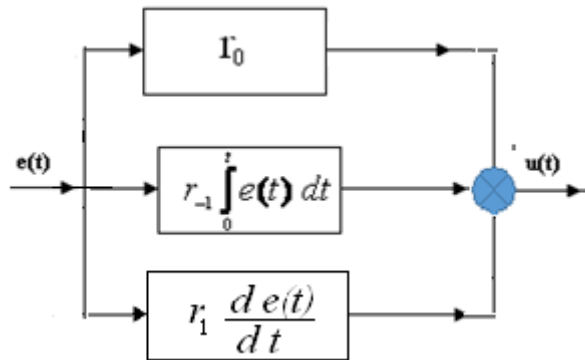
In practice, we cannot build a device that provides ideal derivative action. Commercial PD-controllers are designed on the basis of a **lead-leg** element with very small time constant $\alpha.r_1$.

$$G_r = \frac{r_0 + r_1 p}{\alpha r_1 p + 1}$$

General features of PD-controller

- r_1 is the derivative time constant
- In contrast, proportional and integral controllers are based on the present and past error. Derivative controller action is based on how fast the error is changing in time.
- PD-controller is not useful for systems with large dead time or noisy signals.
- PD control may improve system response while reducing oscillations and overshoot.
- If simple proportional control works fine, we may try PD control. The additionally stabilizing action allows us to use larger proportional gain and obtain a **faster system response**.
- Accelerates system response.

Proportional-Integral-Derivative: PID-controller

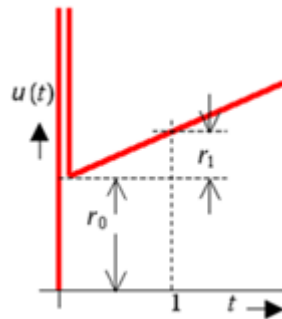


Differential equation of ideal PID-controller is: $u(t) = r_0 e(t) + r_{-1} \int_0^t e(t) dt + r_1 \frac{de(t)}{dt}$

Transfer function of ideal PID-controller is: $U(p) = r_0 E(p) + r_{-1} \frac{1}{p} E(p) + r_1 p E(p) \Rightarrow$

$$G_r(p) = \frac{U(p)}{E(p)} = r_0 + \frac{r_{-1}}{p} + r_1 p = \frac{r_0 p + r_{-1} + r_1 p^2}{p}$$

Step response:
(ideal)



General features of PID-controller

Step response:

With higher-order polynomial in the numerator, the PID-controller is not considered physically realizable. We can say the same as we said about the ideal PD-controller too.

We have to attach the term in denominator again. Small time constant does not influence desired dynamic of the system a lot.

$$\frac{r_0 p + r_{-1} + r_1 p^2}{p(\alpha r_1 p + 1)}$$

- PID controllers eliminate steady state error.
- PID controllers ensure faster acceleration, shorter settling time and smaller overshoot.

Design of regulator.

We introduce feedback to:

- decrease the sensitivity of the system to plant variations
- enable adjustment of the system transient response
- reject disturbances
- reduce steady-state tracking errors

Primary concerns of control system are stability and performance.

Performance is an issue for stable systems.

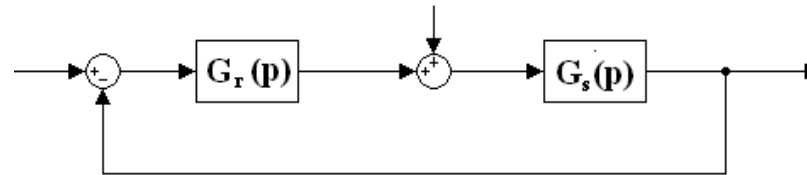
Performance specification can be presented in time domain or the frequency domain.

Time domain specification generally take the stability, form of settling time, peak time, percent overshoot, rise time, and steady state error.

The standard performance measures are usually defined in terms of the step response.

Revision

- Name the main signals in the following control system.



- Explain the principle of this control system.
- Write the formulae of:
 - Control transfer function
 - Deviation (Error) transfer function
 - Disturbance transfer function
- Write the formula for the calculation of steady state error if the input is the unit step.
- Write the formula for the calculation of steady state output if the input is the unit step.
- Write the transfer functions of:
 - P-controller
 - I-controller
 - D-controller. (Is it possible to use D-controller by itself?)
 - PI-controller
 - Ideal PD-controller, real PD-controller
 - Ideal PID-controller, real PID-controller
- Describe general features of P, I, D, PI, PD, PID controllers.
- How do the controllers influence the control process?

References

- [1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.
ISBN: 81-7409-107-6
- [2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing
- [3] Matlab&Simulink: R2015a

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9. Stability Analysis of Closed-loop Control System

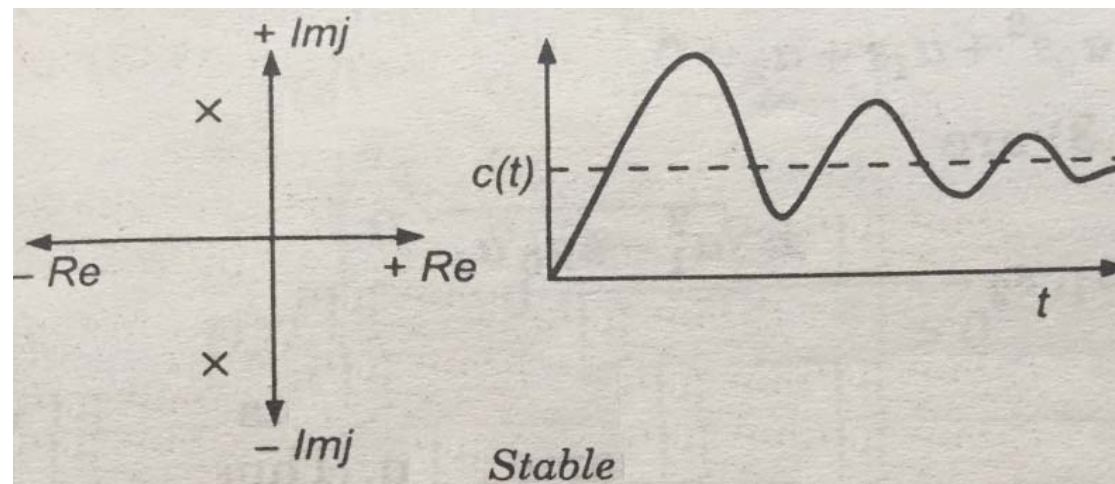
Definition of Stability

- BIBO stability: The system is **Bounded-Input Bounded-Output** stable if the output response is bounded for any bounded input, otherwise the system is said to be not BIBO stable.

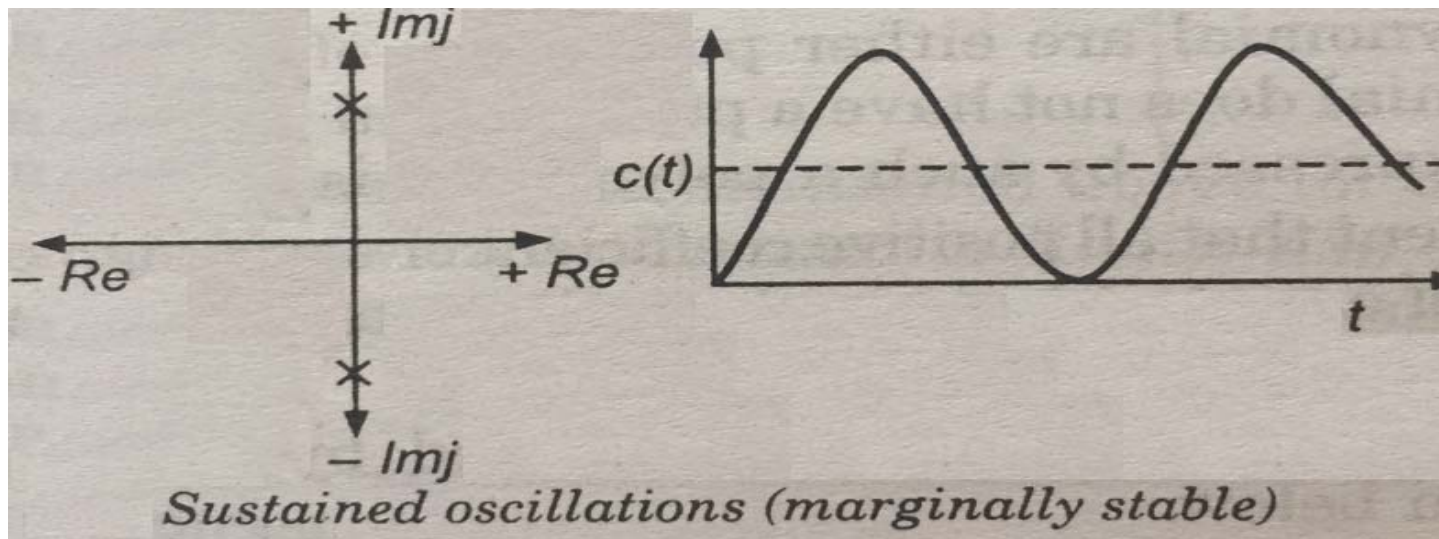
- Stability in terms of characteristic polynomial of a control system:

If the roots of a characteristic equation have negative real part the output response is finite indicating stable system. The roots with positive real part lead to an infinite output response indicating unstable system.

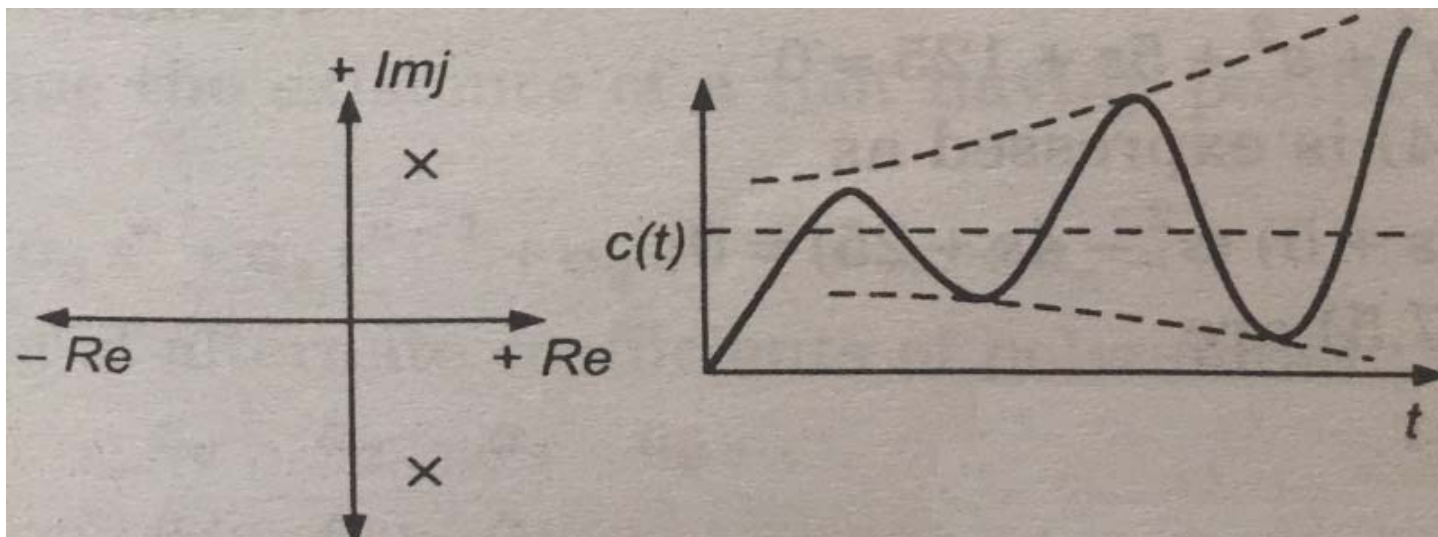
- a) Stable response



- b) Sustained oscillations



- c) Unstable



Methods of stability investigation

- Finding all roots of the characteristic polynomial (if it is possible)
- Using higher-order algebraic methods (Hurwitz, Routh-Schur criterion)
- Root locus method
- Using methods in frequency domain (Bode, Nyquist)

Necessary condition of stability: (but not sufficient)

Consider the system with characteristic equation:

$$a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0 = 0$$

1. Condition – no coefficient is zero, no term is missing.
2. Condition – all coefficients must have the same sign (positive or negative)=>
then the system may or may not be stable.

If one coefficient of the characteristic equation is missing or is not of the same sign =>
the system is unstable.

For polynomial of degree 1 or 2 with a positive leading coefficient, the necessary conditions are sufficient as well. For polynomial of degree 3 or higher we need to use other criterion.

Necessary condition of stability: (but not sufficient)

Using necessary condition of stability determine stability of the systems done by characteristic polynomials

- $A(p) = p^6 + 7p^4 + 17p^2 + 9$
- $A(p) = p^3 + 2p^2 + 9p + 68$
- $A(p) = p^4 + 2p^3 - 4p^2 + 10p - 1$
- $A(p) = 4p^2 + 8p + 4$
- $A(p) = -4p^2 - 8p - 4$

Examples:

1. Consider third order characteristic equation: $p^3+4p^2+6p+4=0$

On factorization is expressed as: $(p+2)(p^2+2p+2)=0$

The three roots are: $p_1=-2$

$$p_2=-1+i$$

$$p_3=-1-i$$

It is observed that none of the roots have positive real parts, therefore the system is stable

- 2. Consider third order characteristic equation: $p^3+p^2+5p+125=0$

On factorization is expressed as: $(p+5)(p^2-4p+25)=0$

The three roots are: $p_1=-5$

$$p_2=+2 + 4,5i$$

$$p_3=+2 - 4,5i$$

Although all the coefficients are positive the equation has distinctly two roots p_2 and p_3 having positive real parts which makes the system unstable.

Hurwitz criterion

In order to determine the existence of a root having positive real part for a n th order polynomial with the positive coefficients can be used algebraic criteria of stability.

For given n th order polynomial $A(p) = a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0$

Hurwitz determinant:
$$H = \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \dots & 0 \\ a_n & a_{n-2} & a_{n-4} & \dots & 0 \\ 0 & a_{n-1} & a_{n-3} & \dots & \vdots \\ \vdots & \vdots & \vdots & & 0 \\ 0 & 0 & 0 & \dots & a_0 \end{bmatrix}$$

For the roots with only negative real parts it is required that the Hurwitz subdeterminants be greater than 0.

$$H_1 = |a_{n-1}| > 0$$

$$H_2 = \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} > 0$$

$$\vdots$$

$$H_n > 0$$

Solved example

System is given by transfer function: $G(p) = \frac{3}{p^4 + 11p^3 + 33p^2 + 37p + 14}$

Hurwitz determinant: $H = \begin{bmatrix} 11 & 37 & 0 & 0 \\ 1 & 33 & 14 & 0 \\ 0 & 11 & 37 & 0 \\ 0 & 1 & 33 & 14 \end{bmatrix}$

Hurwitz subdeterminants: $H_1 = 11 > 0$

$$H_2 = \begin{vmatrix} 11 & 37 \\ 1 & 33 \end{vmatrix} = 326 > 0$$

$$H_3 = \begin{vmatrix} 11 & 37 & 0 \\ 1 & 33 & 14 \\ 0 & 11 & 37 \end{vmatrix} = 10386 > 0$$

$$H_4 = H_3 \cdot 14 > 0$$

All Hurwitz subdeterminants are positive. System is stable. Find the roots in Matlab.

Routh-Schur Criterion

For given n^{th} order polynomial $A(p) = a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0$

1. The coefficients are written into the row starting with the leading coefficient.
2. The coefficients at the even position are underlined and written down into the second row.
3. We reduce the order of characteristic polynomial by one term.
4. The steps 1- 3 are repeated while all coefficients stay positive. If there is a negative coefficient in the row, the system is unstable.
5. The reduction process continues up to a row with three coefficients.
6. If these three coefficients are all positive, the system is stable, otherwise it is unstable.

a_n	<u>a_{n-1}</u>	a_{n-2}	<u>a_{n-3}</u>	a_{n-4}	<u>a_{n-5}</u>	\dots	\dots
<u>a_{n-1}</u>	0	<u>a_{n-3}</u>	0	<u>a_{n-5}</u>	0	\dots	\dots
<hr/>							
$a_n - a_{n-1} \frac{a_n}{a_{n-1}}$	a_{n-1}	$a_{n-2} - a_{n-3} \frac{a_n}{a_{n-1}}$	a_{n-3}	$a_{n-4} - a_{n-5} \frac{a_n}{a_{n-1}}$	a_{n-5}		

Solved example

System is given by transfer function: $G(p) = \frac{3}{p^4 + 11p^3 + 33p^2 + 37p + 14}$

1	11	33	37	14	$\alpha_0 = \frac{1}{11}$
11	0	37	0	0	
$1 - 11 \cdot \alpha_0$	11	$33 - 37 \cdot \alpha_0$	37	14	$\alpha_1 = \frac{11}{\frac{326}{11}} = \frac{121}{326}$
0	11	$\frac{326}{11}$	37	14	
0	$\frac{326}{11}$	0	14	0	
0	0	$\frac{326}{11}$	$\frac{5184}{163}$	14	

Conclusion: All three last coefficients are positive, the system is stable.

Problems

Using Hurwitz or Routh-Schur criterion determine the stability of the systems done by characteristic polynomial.

- $A(p) = p^3 + 2p^2 + 9p + 68$
- $A(p) = 2p^4 + p^3 + 5p^2 + 3p + 4$
- $A(p) = 2p^4 + 5p^3 + 5p^2 + 2p + 1$
- $A(p) = -p^5 - 3p^4 - 10p^3 - 12p^2 - 7p - 3$

Confirm results using the Matlab function “roots”.

Problem solved

Find the ultimate gain r_0 for stable closed-loop system. If we have only a proportional controller $G_r = r_0$ and system given by the transfer function:

$$G_s = \frac{1}{(p+3)(p+2)(p+1)}$$

Control transfer function:
$$G_w(p) = \frac{Y(p)}{W(p)} = \frac{G_r G_s}{1 + G_r G_s} = \frac{\frac{r_0}{(p+3)(p+2)(p+1)}}{1 + \frac{r_0}{(p+3)(p+2)(p+1)}} = \frac{r_0}{(p+3)(p+2)(p+1) + r_0} = \frac{r_0}{p^3 + 6p^2 + 11p + (6+r_0)}$$

Characteristic polynomial: $A(p) = p^3 + 6p^2 + 11p + (6+r_0)$

Hurwitz determinant:
$$H = \begin{vmatrix} 6 & 6+r_0 & 0 \\ 1 & 11 & 0 \\ 0 & 6 & 6+r_0 \end{vmatrix}$$

Hurwitz subdeterminants:
$$H_1 = 6 > 0$$
$$H_2 = \begin{vmatrix} 6 & 6+r_0 \\ 1 & 11 \end{vmatrix} = 66 - 6 - r_0 = 60 - r_0 > 0$$

$$H_3 = H_2(6+r_0) > 0$$

$r_0 < 60 \quad \text{and} \quad r_0 > -6 \quad \rightarrow \quad r_0 \in (-6; 60)$

Direct-Substitution Analysis

The closed-loop poles may lie on the imaginary axis at the moment a system becomes unstable. We can substitute $p = j\omega$ into the closed-loop characteristic equation to find the proportional gain that correspond to this stability limit (which can be called marginal unstable). The value of this specific proportional gain is called the **critical** or **ultimate gain**.

The corresponding frequency is called the **crossover** or **ultimate frequency**.

Example: Apply direct substitution to the characteristic equation from previous example.

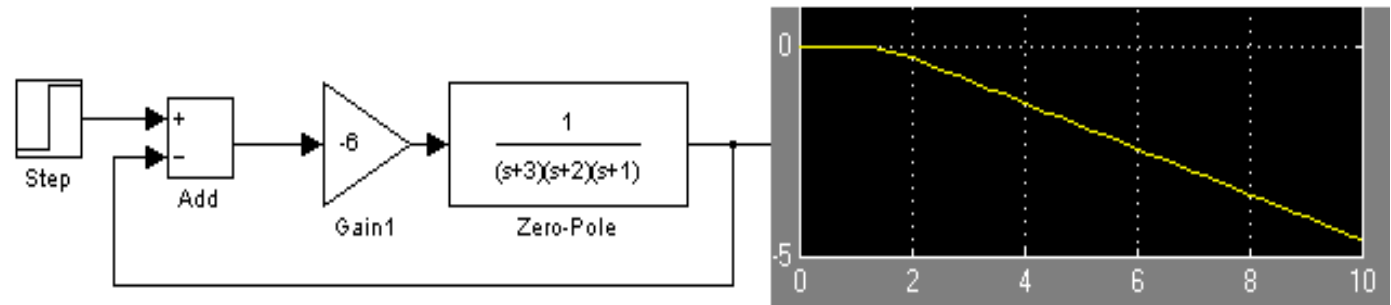
$$p^3 + 6p^2 + 11p + (6 + r_o) = 0$$

- substitution of $p = j\omega$ leads to: $-j\omega^3 - 6\omega^2 + 11j\omega + (6 + r_o) = 0$
- the real and imaginary part of the equation are:
$$\begin{aligned} \text{Re:} \quad & -6\omega^2 + (6 + r_o) = 0 \\ \text{Im:} \quad & -\omega^3 + 11\omega = 0 \end{aligned}$$
- From the imaginary-part of the equation, the ultimate frequency is $\omega_u = \sqrt{11}$
- Substituting this value into the real-part equation leads to the ultimate gain $r_{ou} = 60$, which is consistent with the result of the previous example.
- If we chose the other possibility, $\omega_u = 0$, i.e. the poles are on the real axis, the ultimate gain would be $r_{ou} = -6$, which is also consistent with the result of the previous example.

Simulation in Matlab

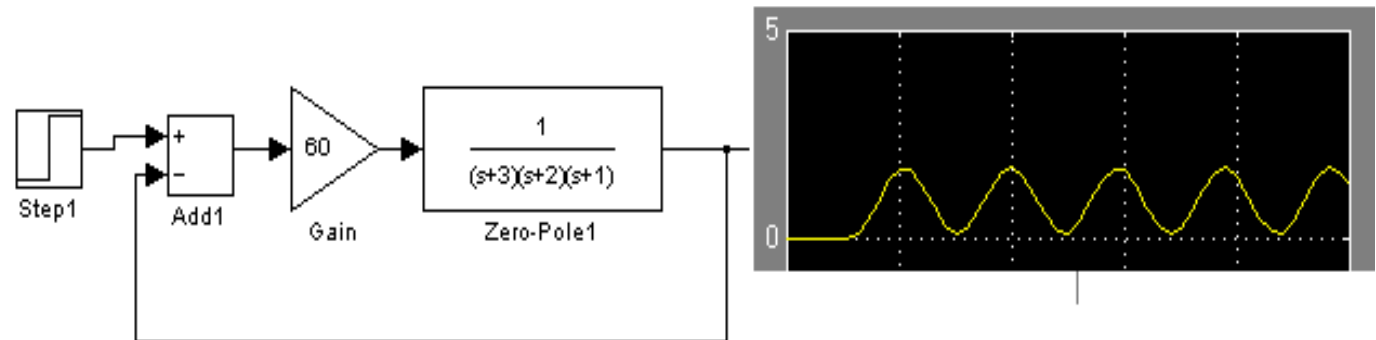
- a) Poles: $0.0000 + 0.0000i$
 $-3.0000 + 1.4142i$
 $-3.0000 - 1.4142i$

System unstable.



- b) Poles: $-6.0000 + 0.0000i$
 $-0.0000 + 3.3166i$
 $-0.0000 - 3.3166i$

System on stability limit:
 sustained oscillation



Problem solved

Assume the second-order transfer function. If we synthesize the control with PI controller, what are the stability constraints?

$$G_s = \frac{1}{p^2 + 2p + 1} \quad G_r = r_0 + \frac{r_{-1}}{p} = \frac{r_0 p + r_{-1}}{p}$$

$$G_w = \frac{Y(p)}{W(p)} = \frac{G_r \cdot G_s}{1 + G_r \cdot G_s} = \frac{\frac{r_0 p + r_{-1}}{p} \cdot \frac{1}{p^2 + 2p + 1}}{1 + \frac{r_0 p + r_{-1}}{p(p^2 + 2p + 1)}} = \frac{r_0 p + r_{-1}}{p^3 + 2p^2 + (1 + r_0)p + r_{-1}}$$

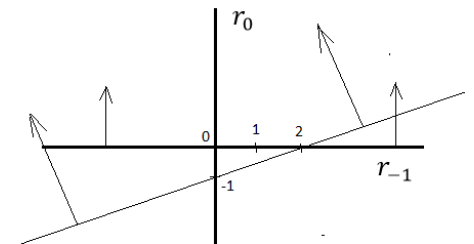
The characteristic equation is: $p^3 + 2p^2 + (1 + r_0)p + r_{-1} = 0$

$$H = \begin{vmatrix} 2 & r_{-1} & 0 \\ 1 & 1 + r_0 & 0 \\ 0 & 2 & r_{-1} \end{vmatrix}$$

$$H_1 = 2 > 0$$

$$H_2 = 2(1 + r_0) + r_{-1} > 0 \Rightarrow 2r_0 > r_{-1} - 2 \Rightarrow r_0 > \frac{1}{2}r_{-1} - 1$$

$$\text{or } r_{-1} < 2r_0 + 2$$



Problems

1. Find the stability constraints.

$$G_s = \frac{1}{a_2 p^2 + a_1 p} \quad a_2, a_1 > 0$$

$$a) \quad G_R = \frac{r_{-1}}{p}$$

$$b) \quad G_R = r_0 + \frac{r_{-1}}{p}$$

2. Characteristic equation is: $A(p) = 0,002p^3 + 0,08p^2 + p(0,15K - 1) + K$
Find the condition for K for the system to be unstable.

Nyquist criterion

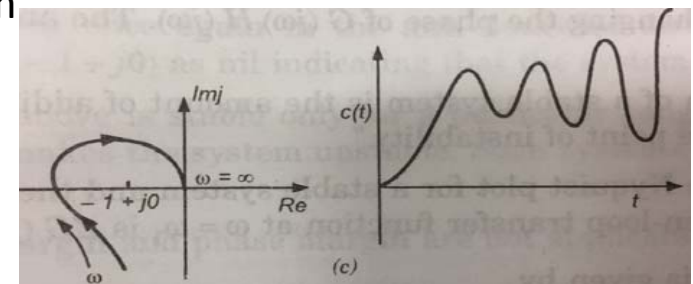
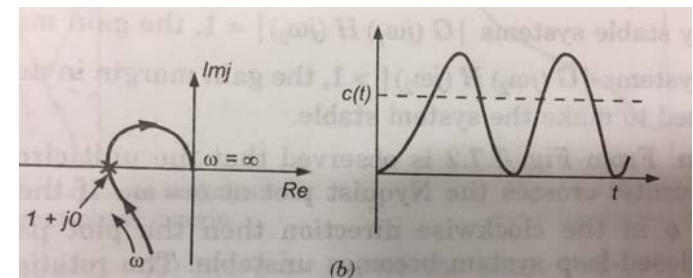
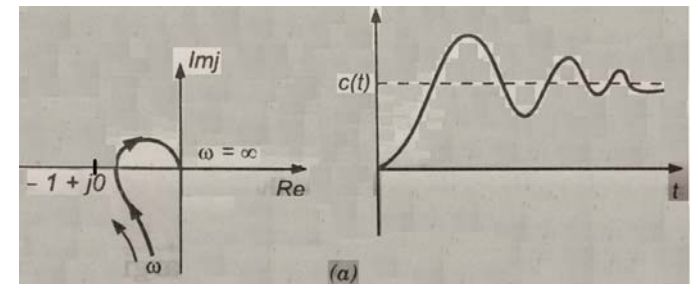
Application of Nyquist criterion to determine stability of closed loop control system.

Closed loop stability is determined from Nyquist plot of open-loop transfer function: $G_o(p)=G_s(p) \cdot G_r(p)$.

The Nyquist plot for a stable system is shown in Fig. (a). The plot crosses the negative real axis at ω when the corresponding magnitude $|G_o(j\omega)|$ is less than 1.

(b) If the plot passes through the point $(-1+j0)$ the system starts tending towards instability. The magnitude of $|G_o(j\omega)|$ equals to 1 at this point.

(c) It is concluded that if the gain is increased the closed-loop system becomes unstable.



Nyquist criterion

Gain margin: Nyquist plot gives the information about the stability and also indicates how far the system is stable, how the performance of the system can be improved in view of stability.

The **gain margin** is a factor by which the gain of a stable system is allowed to increase before the system reaches instability.

In decibel the gain margin is given by: $G.M. = 20 \log_{10} \frac{1}{|G_0(j\omega)|} \text{ (db)}$

For stable systems $|G_0(j\omega)| < 1$, the gain margin in decibel is positive.

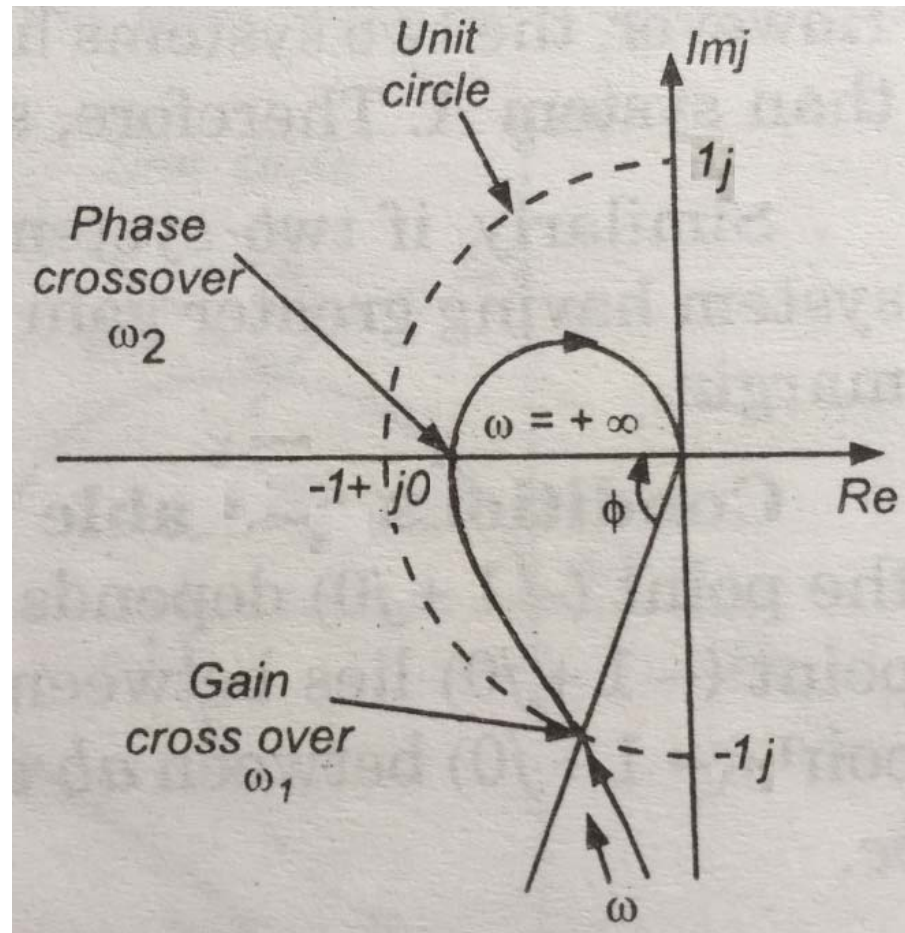
For marginally stable systems $|G_0(j\omega)| = 1$, the gain margin in decibel is zero.

For unstable systems $|G_0(j\omega)| > 1$, the gain margin in decibel is negative and the gain is to be reduced to make the system stable.

Phase margin: Is observed that the unit circle crosses the Nyquist plot at ω .

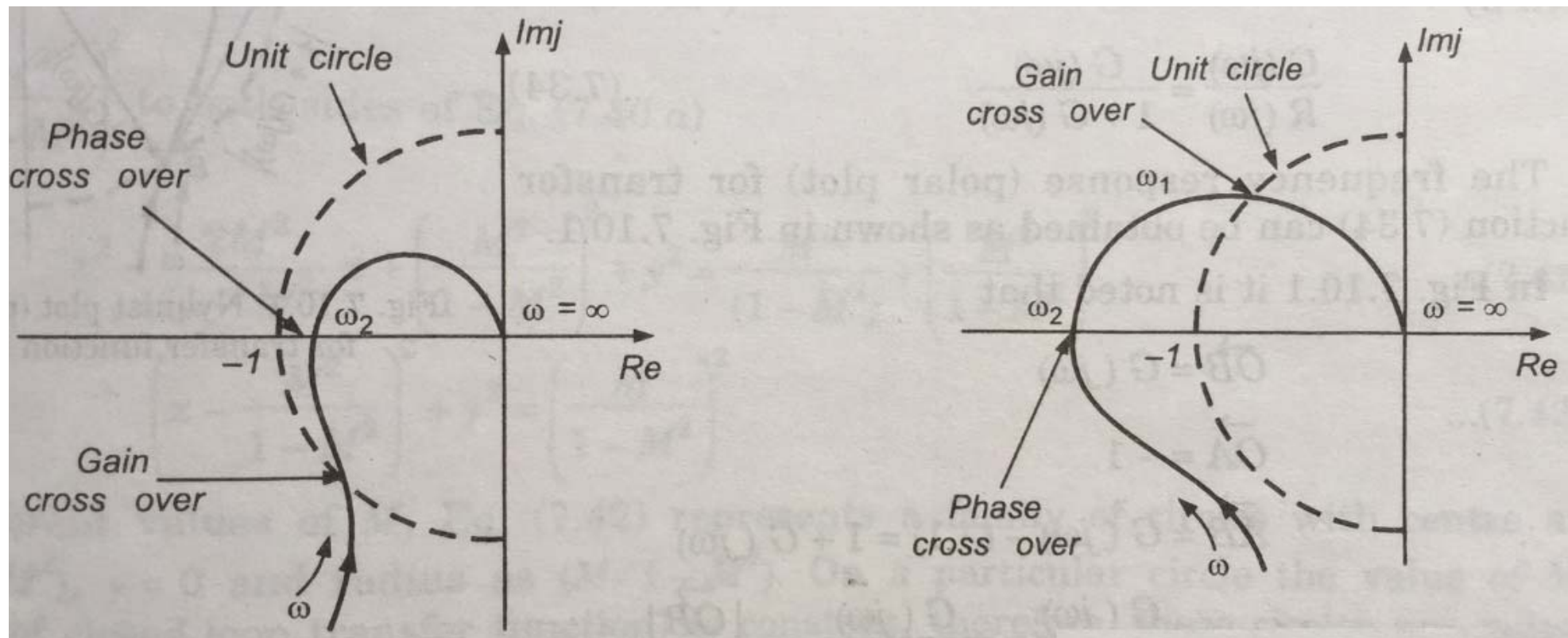
The phase margin of a stable system is the amount of additional phase lag required to bring the system to the point of instability. The phase margin is given by $P.M. = 180^\circ + \arctan|G_0(j\omega)|$

Gain margin and phase margin from Nyquist plot



Nyquist plot for stable system and for unstable system

On the assumption that there is none unstable pole in the system $G_s(j\omega)$.



Solved example: Nyquist criterion

Examine the closed-loop stability of the system whose open-loop transfer function is given by:

$$G_o(p) = \frac{50}{(p+1)(p+2)}$$

Using Matlab:

```
num =
```

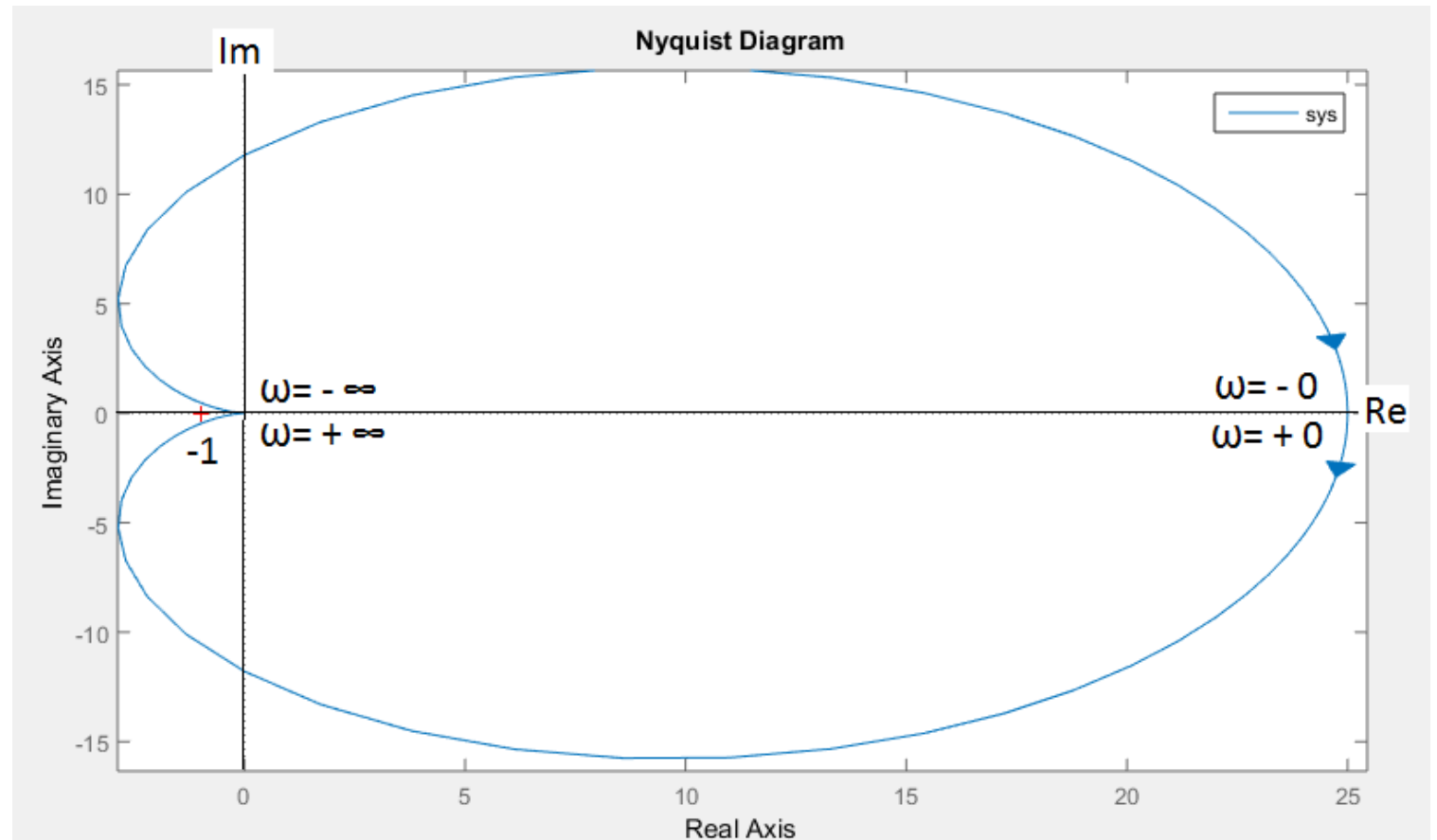
```
50
```

```
>> den=[1 3 2]
```

```
den =
```

```
1 3 2
```

```
>> nyquist(num,den)
```



Determination gain margin and phase margin in Bode plot

The gain margin and phase margin can also be obtained from the Bode plot of the open-loop transfer function and the stability of closed-loop system can be determined.

Illustrative example: $|G_0(p)| = \frac{4}{p(0.5p + 1)(0.08p + 1)}$

The corner frequencies are: $\omega_1 = \frac{1}{0.5} = 2 \text{ rad/sec}$ $\omega_2 = \frac{1}{0.08} = 12.5 \text{ rad/sec}$

The starting frequency of the Bode plot is taken as lower than the lowest corner frequency: starting frequency is taken as 1 rad/sec. The initial magnitude for $\omega=1$ is $|A(\omega=1)|_{\text{db}} = 20\log(4/1) = 12\text{db}$.

The initial part of the Bode plot is -20 db/decade.

Phase for frequencies between $\omega=1$ rad/sec to $\omega=100$ rad/sec is calculated as below:

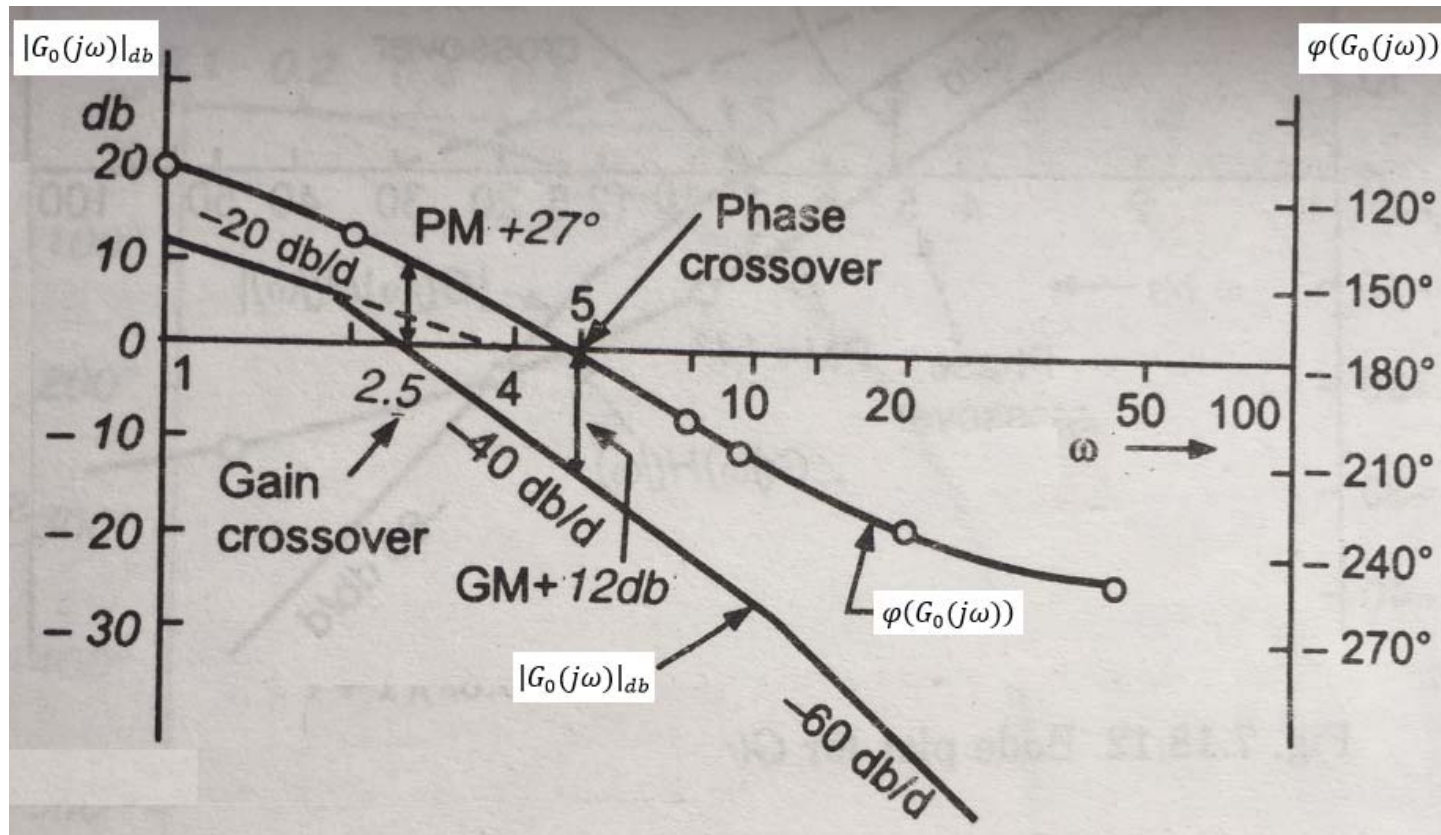
$$\varphi(G_0(j\omega)) = -90^\circ - \arctan(0.5\omega) - \arctan(0.08\omega)$$

The gain in db at phase cross over frequency ($\omega=5$) is the gain margin: G.M. = 12 db.

The gain cross over frequency is 2.5 rad/sec, the phase $\varphi(\omega=2.5) = -153^\circ$.

The phase margin is given: P.M. = $180^\circ - \varphi(\omega=2.5) = 180^\circ + (-153^\circ) = 27^\circ$

Bode plot for $|G_0(p)| = \frac{4}{p(0.5p + 1)(0.08p + 1)}$



The gain margin and phase margin both are positive, therefore, the closed-loop system is stable.

Revision questions

- Which of components of PID:
 - a) can accelerate the response of the control system
 - b) can eliminate the error-steady state value
 - c) can extend the settling time
- Where are the poles of the transfer function located, if
 - the system is stable
 - the system is unstable
 - the system is oscillating with constant amplitude
- The necessary condition of stability.
- Which algebraic criteria of stability do you know?

References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.

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10. Quality of Control Performance.

Quality of Control Performance

- Stability
- Control performance quality (quality criteria)
 - Static accuracy - steady-state deviation
 - Settling time
 - Aperiodicity criterion
 - Integral criterion

Steady-state deviation for different types of systems, different types of controllers and different types of input signal.

1) Types of the system:

- Static system: no root in origin ($s=0$)
- Astatic system: one or more roots in origin ($s= 1, 2, \dots$)

2) Type of the controllers:

- P (r=0)
- I (r=1)
- PI (r=1)
-
- PD (r=0)
- PID (r=1)

3) Types of input signal

a) Control input signal

Constant control: $w(t) = w_0$

Linear increasing control: $w(t) = w_1(t)$

b) Disturbance input signal

Constant disturbance: $z(t) = z_0$

Linear increasing disturbance: $z(t) = z_1(t)$

Steady-state value of deviation if input signal is W(p)

Transfer function of deviation: $G_e(p) = \frac{1}{1 + G_r(p) \cdot G_s(p)}$

Error signal: $E(p) = G_e(p) \cdot W(p) = \frac{1}{1 + G_r(p) \cdot G_s(p)} \cdot W(p)$

Steady state value of error signal: $e(\infty) = \lim_{p \rightarrow 0} p \cdot E(p)$

$r + s \backslash W$	$\frac{w_0}{p}$	$\frac{w_1}{p^2}$
0	$e(\infty) = \frac{w_0}{1+r_0} \neq 0$	$e(\infty) = \infty$
1	$e(\infty) = 0$	$e(\infty) = \frac{w_1}{r_0} \neq 0$
2	$e(\infty) = 0$	$e(\infty) = 0$

Solved example

Determine steady state deviation. Input signal is $W(p)$.

1. $s = 0, r = 0:$ $G_s = \frac{1}{p^2 + 2p + 1}$ $G_R = r_0$ $W(p) = \frac{w_0}{p}$

$$G_e = \frac{E(p)}{W(p)} = \frac{1}{1 + G_s G_R} = \frac{1}{1 + \frac{r_0}{p^2 + 2p + 1}} = \frac{p^2 + 2p + 1}{p^2 + 2p + (1 + r_0)}$$

$$E(p) = G_e \frac{w_0}{p}$$

$$e(\infty) = \lim_{p \rightarrow 0} p \cdot G_e \cdot \frac{w_0}{p} = \frac{w_0}{1 + r_0}$$

Solved example

Determine steady state deviation. Input signal is $W(p)$.

$$2. \boxed{s = 0, r = 1:} \quad G_s = \frac{1}{p^2 + 2p + 1} \quad G_r = \frac{r_{-1}}{p} \quad W(p) = \frac{w_0}{p}$$

$$G_e = \frac{E(p)}{W(p)} = \frac{1}{1 + G_s G_r} = \frac{1}{1 + \frac{1}{p^2 + 2p + 1} \cdot \frac{r_{-1}}{p}} = \frac{p \cdot (p^2 + 2p + 1)}{p^3 + 2p^2 + p + r_{-1}}$$

$$E(p) = G_e \frac{w_0}{p}$$

$$e(\infty) = \lim_{p \rightarrow 0} p \cdot G_e \cdot \frac{w_0}{p} = \lim_{p \rightarrow 0} p \cdot \frac{p \cdot (p^2 + 2p + 1)}{p^3 + 2p^2 + p + r_{-1}} \cdot \frac{w_0}{p} = 0$$

Solved example

Determine steady state deviation. Input signal is $W(p)$.

3. $s = 0, r = 1:$ $G_s = \frac{1}{p^2 + 2p + 1}$ $G_R = r_0 + \frac{r_{-1}}{p}$ $W(p) = \frac{w_0}{p}$

$$G_e = \frac{E(p)}{W(p)} = \frac{1}{1 + G_s G_R} = \frac{1}{1 + \frac{1}{p^2 + 2p + 1} \cdot \frac{r_0 p + r_{-1}}{p}} = \frac{p \cdot (p^2 + 2p + 1)}{p^3 + 2p^2 + p(1 + r_0) + r_{-1}}$$

$$E(p) = G_e \frac{w_0}{p}$$

$$e(\infty) = \lim_{p \rightarrow 0} p \cdot G_e \cdot \frac{w_0}{p} = \lim_{p \rightarrow 0} p \cdot \frac{p \cdot (p^2 + 2p + 1)}{p^3 + 2p^2 + p(1 + r_0) + r_{-1}} \cdot \frac{w_0}{p} = 0$$

Solved example

Determine steady state deviation. Input signal is $W(p)$.

4. $s = 0, r = 1:$ $G_s = \frac{1}{p^2 + 2p + 1}$ $G_R = \frac{r_{-1}}{p}$ $W(p) = \frac{w_1}{p^2}$

$$G_e = \frac{E(p)}{W(p)} = \frac{1}{1 + G_s G_r} = \frac{1}{1 + \frac{1}{p^2 + 2p + 1} \cdot \frac{r_{-1}}{p}} = \frac{p \cdot (p^2 + 2p + 1)}{p^3 + 2p^2 + p + r_{-1}}$$

$$E(p) = G_e \frac{w_1}{p^2}$$

$$e(\infty) = \lim_{p \rightarrow 0} p \cdot G_e \cdot \frac{w_1}{p^2} = \lim_{p \rightarrow 0} p \cdot \frac{p \cdot (p^2 + 2p + 1)}{p^3 + 2p^2 + p + r_{-1}} \cdot \frac{w_1}{p^2} = \frac{w_1}{r_0}$$

Conclusion

Condition for zero steady-state of deviation:

$$s + r \geq 1 + l$$

Where:

- s = number of roots in origin in transfer function of system
- r = number of roots in origin in transfer function of controller
- $l = 0$ when control input is step function
- $l = 1$ when control input is ramp function

Steady-state value depends on the sum of astatic poles, no matter if they belong to the system or to the controller.

Steady-state value of deviation if input signal is Z(p)

Transfer function of disturbance: $G_z(p) = \frac{Y(p)}{Z(p)} = \frac{G_s(p)}{1 + G_s(p) \cdot G_r(p)}$

Output signal: $Y(p) = \frac{G_s(p)}{1 + G_s(p) \cdot G_r(p)} \cdot Z(p)$

Steady state value of error signal: $e(\infty) = -y(\infty) = -\lim_{p \rightarrow 0} p \cdot G_z(p) \cdot Z(p)$

$\begin{array}{c} Z \\ r+s \end{array}$	$\frac{z_0}{p}$	$\frac{z_1}{p^2}$
$r=0, s=0$	$e(\infty) = -\frac{z_0}{1+r_0}$	$e(\infty) = -\infty$
$r=0, s>0$	$e(\infty) = -\frac{z_0}{r_0}$	$e(\infty) = -\infty$
$r=1, s=0$	$e(\infty) = 0$	$e(\infty) = -\frac{z_1}{r_{-1}}$
$r=1, s>0$	$e(\infty) = 0$	$e(\infty) = -\frac{z_1}{r_{-1}}$

Solved example

Determine steady state deviation. Input signal is disturbance $Z(p)$.

1. $s = 0, r = 0$: $G_s = \frac{1}{p^2 + 2p + 1}$ $G_R = r_0$ $Z(p) = \frac{z_0}{p}$

$$G_z = \frac{Y(p)}{Z(p)} = \frac{G_s}{1 + G_s G_R} = \frac{\frac{1}{p^2 + 2p + 1}}{1 + \frac{1}{p^2 + 2p + 1} \cdot r_0} = \frac{1}{p^2 + 2p + 1 + r_0}$$

$$Y(p) = G_z \cdot \frac{z_0}{p}$$

$$(\infty) = -y(\infty) = -\lim_{p \rightarrow 0} p \cdot G_z \cdot \frac{z_0}{p} = -\lim_{p \rightarrow 0} p \cdot \frac{1}{p^2 + 2p + 1 + r_0} \cdot \frac{z_0}{p} = -\frac{z_0}{1 + r_0}$$

Solved example

Determine steady state deviation. Input signal is disturbance $Z(p)$.

2. $s=0, r=0$: $G_s = \frac{1}{p^2 + 2p + 1}$ $G_R = r_0$ $Z(p) = \frac{z_1}{p^2}$

$$G_z = \frac{Y(p)}{Z(p)} = \frac{G_s}{1 + G_s G_R} = \frac{\frac{1}{p^2 + 2p + 1}}{1 + \frac{1}{p^2 + 2p + 1} \cdot r_0} = \frac{1}{p^2 + 2p + 1 + r_0}$$

$$Y(p) = G_z \cdot \frac{z_1}{p^2}$$

$$e(\infty) = -y(\infty) = -\lim_{p \rightarrow 0} p \cdot G_z \cdot \frac{z_1}{p^2} = -\lim_{p \rightarrow 0} p \cdot \frac{1}{p^2 + 2p + 1 + r_0} \cdot \frac{z_1}{p^2} = -\infty$$

Solved example

Determine steady state deviation. Input signal is disturbance $Z(p)$.

3. $s=1, r=0$: $G_s = \frac{1}{p \cdot (p+1)}$ $G_R = r_0$ $Z(p) = \frac{z_0}{p}$

$$G_z = \frac{Y(p)}{Z(p)} = \frac{G_s}{1 + G_s G_r} = \frac{\frac{1}{p \cdot (p+1)}}{1 + \frac{1}{p \cdot (p+1)} \cdot r_0} = \frac{1}{p^2 + p + r_0}$$

$$Y(p) = G_z \cdot \frac{z_0}{p}$$

$$(\infty) = -y(\infty) = -\lim_{p \rightarrow 0} p \cdot G_z \cdot \frac{z_0}{p} = -\lim_{p \rightarrow 0} p \cdot \frac{1}{p^2 + p + r_0} \cdot \frac{z_0}{p} = -\frac{z_0}{r_0}$$

Solved example

Determine steady state deviation. Input signal is disturbance $Z(p)$.

4. $s=0, r=1:$ $G_s = \frac{1}{p^2+2p+1}$ $G_R = r_0 + \frac{r_{-1}}{p}$ $Z(p) = \frac{z_0}{p}$

$$G_z = \frac{Y(p)}{Z(p)} = \frac{G_s}{1 + G_s G_R} = \frac{\frac{1}{p^2 + 2p + 1}}{1 + \frac{1}{p^2 + 2p + 1} \cdot \frac{r_0 p + r_{-1}}{p}} = \frac{p}{p^3 + 2p^2 + p(1 + r_0)r_{-1}}$$

$$Y(p) = G_z \cdot \frac{z_0}{p}$$

$$e(\infty) = -y(\infty) = -\lim_{p \rightarrow 0} p \cdot G_z \cdot \frac{z_0}{p} = -\lim_{p \rightarrow 0} p \cdot \frac{p}{p^3 + 2p^2 + p(1 + r_0)r_{-1}} \cdot \frac{z_0}{p} = 0$$

Solved example

Determine steady state deviation. Input signal is disturbance $Z(p)$.

5. $s = 0, r = 1:$ $G_s = \frac{1}{p^2 + 2p + 1}$ $G_R = r_0 + \frac{r_{-1}}{p}$ $Z(p) = \frac{z_1}{p^2}$

$$G_z = \frac{Y(p)}{Z(p)} = \frac{G_s}{1 + G_s G_R} = \frac{\frac{1}{p^2 + 2p + 1}}{1 + \frac{1}{p^2 + 2p + 1} \cdot \frac{r_0 p + r_{-1}}{p}} = \frac{p}{p^3 + 2p^2 + p(1 + r_0) + r_{-1}}$$

$$Y(p) = G_z \cdot \frac{z_1}{p^2}$$

$$e(\infty) = -y(\infty) = -\lim_{p \rightarrow 0} p \cdot G_z \cdot \frac{z_1}{p^2} = -\lim_{p \rightarrow 0} p \cdot \frac{p}{p^3 + 2p^2 + p(1 + r_0) + r_{-1}} \cdot \frac{z_1}{p^2} = -\frac{z_1}{r_{-1}}$$

Solved example

Determine steady state deviation. Input signal is disturbance $Z(p)$.

6. $s=1, r=1:$ $G_s = \frac{1}{p(p+1)}$ $G_R = r_0 + \frac{r_{-1}}{p}$ $Z(p) = \frac{z_0}{p}$

$$G_z = \frac{Y(p)}{Z(p)} = \frac{G_s}{1 + G_s G_r} = \frac{\frac{1}{p \cdot (p+1)}}{1 + \frac{1}{p \cdot (p+1)} \cdot \frac{r_0 p + r_{-1}}{p}} = \frac{p}{p^2 + p(1 + r_0) + r_{-1}}$$

$$Y(p) = G_z \cdot \frac{z_0}{p}$$

$$(\infty) = -y(\infty) = -\lim_{p \rightarrow 0} p \cdot G_z \cdot \frac{z_0}{p} = -\lim_{p \rightarrow 0} p \cdot \frac{p}{p^2 + p(1 + r_0) + r_{-1}} \cdot \frac{z_0}{p} = 0$$

Solved example

Determine steady state deviation. Input signal is disturbance $Z(p)$.

7. $s=1, r=1:$ $G_s = \frac{1}{p(p+1)}$ $G_R = r_0 + \frac{r_{-1}}{p}$ $Z(p) = \frac{z_1}{p^2}$

$$G_z = \frac{Y(p)}{Z(p)} = \frac{G_s}{1 + G_s G_r} = \frac{\frac{1}{p \cdot (p+1)}}{1 + \frac{1}{p \cdot (p+1)} \cdot \frac{r_0 p + r_{-1}}{p}} = \frac{p}{p^2 + p(1 + r_0) + r_{-1}}$$

$$Y(p) = G_z \cdot \frac{z_1}{p^2}$$

$$e(\infty) = -y(\infty) = -\lim_{p \rightarrow 0} p \cdot G_z \cdot \frac{z_1}{p^2} = -\lim_{p \rightarrow 0} p \cdot \frac{p}{p^2 + p(1 + r_0) + r_{-1}} \cdot \frac{z_1}{p^2} = -\frac{z_1}{r_{-1}}$$

Conclusion

- Condition for zero steady-state of deviation: $s + r \geq 1 + m + \xi$

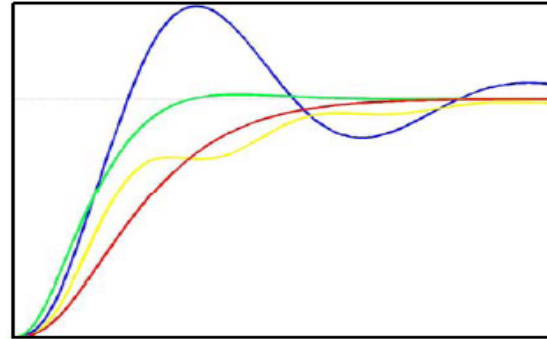
Where:

- s = number of roots in origin in transfer function of system
- r = number of roots in origin in transfer function of controller
- m = number of roots in origin in disturbance transfer function
- $\xi = 0$ when disturbance input is step function
- $\xi = 1$ when disturbance input is ramp function

Aperiodicity criterion

Basic types of transient response:

- Oscillatory response with overshoot
- Oscillatory response without overshoot
- Aperiodic response with overshoot
- Aperiodic response without overshoot



Procedure of aperiodicity test:

- Assume the characteristic polynomial of a stable control loop: $A(p) = a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0$
- Create sequence of $2n+1$ terms: $a_n \quad n \cdot a_n \quad a_{n-1} \quad (n-1)a_{n-1} \quad \dots \quad a_2 \quad 2 \cdot a_2 \quad a_1 \quad a_1 \quad a_0$
- Apply **Routh-Shur** algorithm to reduce the terms step by step to the sequence consisting of 3 terms.
- Determine the type of response:
 - If all 3 coefficients are positive, system has an aperiodic response.
 - If there is a negative coefficient, system has an oscillatory response.
 - If there is a zero coefficient, system has a response on the boundary of aperiodicity.

Aperiodicity criterion: Problem solved

Characteristic polynomial is $A(p) = 4p^2 + kp + 2$

Find k to set the step response on the boundary of aperiodicity.

4	8	k	k	2	$\alpha_0 = \frac{4}{8} = \frac{1}{2}$
8	0	k	0	0	
$4 - \frac{1}{2}8$	8	$k - \frac{k}{2}$	k	2	$\alpha_1 = \frac{8}{\frac{k}{2}} = \frac{16}{k}$
	$\frac{k}{2}$	0	2	0	
	0	$\frac{k}{2}$	$k - \frac{32}{k}$	2	

Condition for boundary of aperiodicity: $k - \frac{32}{k} = 0$

1) $k_1 = +\sqrt{32}$

2) $k_2 = -\sqrt{32}$

Integral Criteria of Control Quality

Integral criteria of control quality search the optimum controller constants by minimization of a proper goal function – optimality criterion.

1. Linear integral criterion IE (linear control area): $I_l = \int_0^{-\infty} e(t)dt \rightarrow \min$

2. Quadratic integral criterion ISE (quadratic control area): integral of the square error:

$$I_q = \int_0^{-\infty} e^2(t)dt \rightarrow \min$$

3. Integral criterion IAE : integral of absolute error:

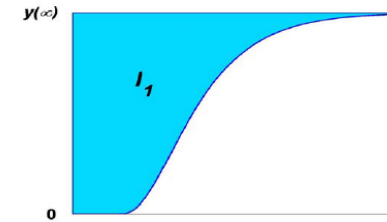
$$I_A = \int_0^{-\infty} |e(t)|dt \rightarrow \min$$

4. Integral criterion ITAE: integral of time weighted absolute error: $I_{ITAE} = \int_0^{-\infty} t \cdot |e(t)|dt \rightarrow \min$

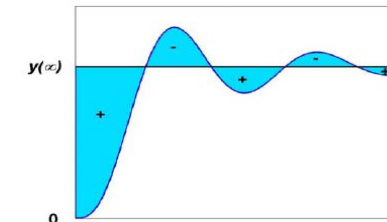
Integral Criteria of Control Quality

Graphic representation:

1. Linear integral criterion IE (linear control area):

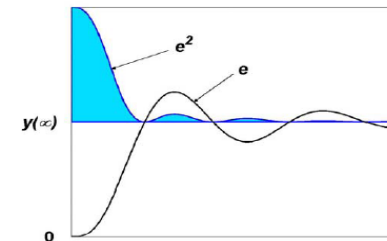


2. Integral criterion IAE : integral of absolute error:



3. Quadratic integral criterion ISE (quadratic control area): integral of the square error:

$$I_q = \int_0^{+\infty} e^2(t) dt = \int_0^{+\infty} [y(\infty) - y(t)]^2 dt \rightarrow \min$$



Disadvantages of use of the quadratic integral criterion:

- Greater deviations (beginning of response) have higher weight during calculation.
- The optimum responses would suppose less deviations → there is a overshoot.
- The optimum control performance is oscillating.

Calculation of the quadratic integral criterion by means of Hurwitz matrix.

- Calculation of control transfer function or deviation transfer function:

$$G_w(p) = \frac{G_s(p)G_r(p)}{1 + G_s(p)G_r(p)} \quad G_e(p) = \frac{1}{1 + G_s(p)G_r(p)}$$

- Laplace transform of deviation: $E(p) = G_e \frac{1}{p} = \frac{b_{n-1}p^{n-1} + b_{n-2}p^{n-2} + \dots + b_1p + b_0}{a_n p^n + a_{n-1}p^{n-1} + \dots + a_1p + a_0}$

- or: $\overline{Y(p)} = Y(p) - \frac{1}{p} \lim_{p \rightarrow 0} p \cdot Y(p)$

- Calculation of Hurwitz determinant:

$$H_n = \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \dots & 0 \\ a_n & a_{n-2} & a_{n-4} & \dots & 0 \\ 0 & a_{n-1} & a_{n-3} & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & a_0 \end{bmatrix}$$

Calculation of the quadratic integral criterion by means of Hurwitz matrix.

- Calculation of modified Hurwitz determinant:

$$H_1 = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} \\ a_n & a_{n-2} & a_{n-4} & \cdots & 0 \\ 0 & a_{n-1} & a_{n-3} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & a_0 \end{bmatrix}$$

$$h_{11} = (-1)^0 b_{n-1}^2$$

$$h_{12} = (-1)^1 (b_{n-2}^2 - 2b_{n-1}b_{n-3})$$

$$h_{13} = (-1)^2 (b_{n-3}^2 - 2b_{n-2}b_{n-4} + 2b_{n-1}b_{n-5})$$

$$\vdots$$

$$h_{1n} = (-1)^{n-1} b_0^2$$

- The quadratic control area is calculated from the following formula:

$$I_q = \frac{1}{2 a_n} \frac{\det(H_1)}{\det(H_n)}$$

Problem

Assume the system given by the transfer function: $G_s = \frac{1}{(10p + 1)^2}$

Three controllers were designed for this system. Compute the quadratic control area and decide which controller is the best according to the quadratic integral criterion. Simulate and compare the step responses in Matlab.

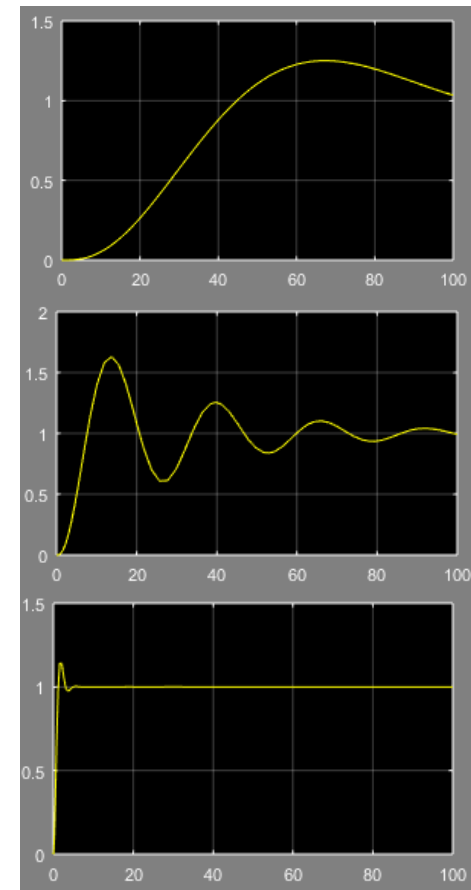
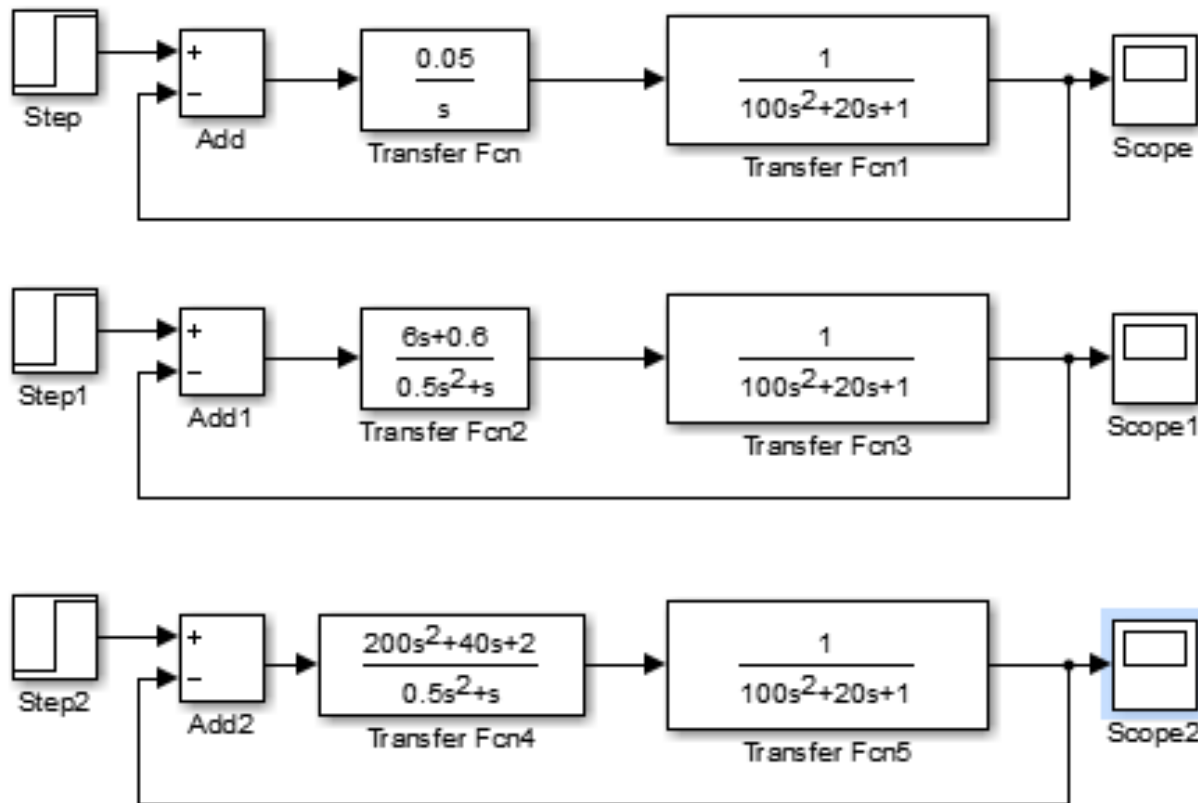
$$G_{R1} = \frac{0,05}{p}$$

$$G_{R2} = \frac{0,6(10p + 1)}{p(0,5p + 1)}$$

$$G_{R3} = \frac{2(10p + 1)^2}{p(0,5p + 1)}$$

Results: 1) $I_q = 23,3$ 2) $I_q = 8,16$ 3) $I_q = 0,5$

Solution: Simulation in Matlab



References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.

ISBN: 81-7409-107-6

[2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing

[3] Matlab&Simulink: R2015a

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11. Design of Controllers

Design of regulators

We introduce feedback to:

- decrease the sensitivity of the control system concerning the changes of the controlled system
- enable adjustment of the system transient response
- reject disturbances
- reduce steady-state tracking errors

Two problems are necessary to be solved at controller applying:

- choice of suitable controller type (to achieve stability and zero steady-state deviation)
- setting controllers' parameters = controller adjustment (to achieve desired settling time, overshoot, and other time parameters of the response)

Usability of controller types:

Controller	System
P	0 th or 1 st order Little dead time Little disturbances It causes steady-state deviation
PI	Higher order system Less dead time Big but not frequent changes of disturbances Zero steady-state deviation
PD	Higher order systems with dead time Little and not frequent changes of disturbances Non-zero steady-state deviation
PID	Higher order system with dead time Big and frequent changes of disturbances Zero steady-state deviation

Design of regulators

A. Empirical methods of controller adjustment:

1. Ziegler-Nichols method
2. Method of standard form of characteristic equation
 - a) ITAE criterion
 - b) Whiteley forms
 - c) Multiple roots adjustment
 - d) Two dominant roots adjustment

B. Standard methods of controller adjustment

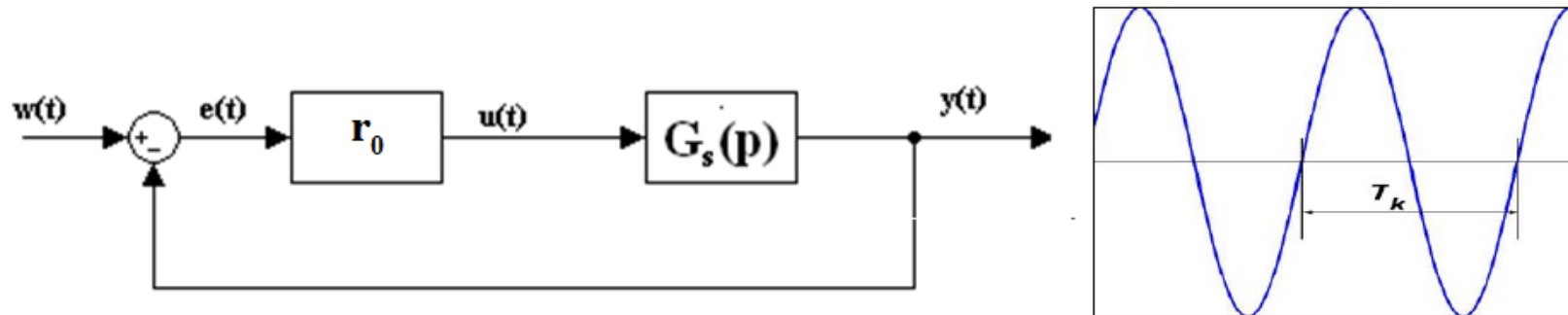
1. Integral criterion
2. Optimum module criterion

C. Design of state-space regulator

1. Root locus technique
2. LQR - methods

Ziegler-Nichols method

- This method is based on closed-loop testing (called on-line tuning), of processes that are inherently stable, but in which the system may become unstable. We use only P-controller in the experiment, the I and D controllers are disabled. Then the proportional gain is slowly increased until the undamped oscillations begin in the system.
- The proportional gain and the period of oscillation at this point are called the **ultimate gain** r_{ok} and **ultimate period** T_k .



- These two quantities are used in a set of empirical tuning relations developed by Ziegler and Nichols, listed in table below:

$$G_R = r_0 + \frac{r_{-1}}{p} + r_1 p = K \left(1 + \frac{1}{T_i p} + T_d p \right)$$

Ziegler-Nichols method

Optimal setting of the controllers

Controller	r_0	r_{-1}	r_1	T_i	T_d
P	$0,5 r_{0k}$	-	-	-	-
PI	$0,45 r_{0k}$	$0,53 r_{0k} / T_k$	-	$0,85 T_k$	-
PD	$0,4 r_{0k}$	-	$0,02 r_{0k} T_k$	-	$0,05 T_k$
PID	$0,6 r_{0k}$	$1,2 r_{0k} / T_k$	$0,072 r_{0k} T_k$	$0,5 T_k$	$0,12 T_k$

$$G_R = r_0 + \frac{r_{-1}}{p} + r_1 p = K \left(1 + \frac{1}{T_i p} + T_d p \right)$$

Ziegler-Nichols method

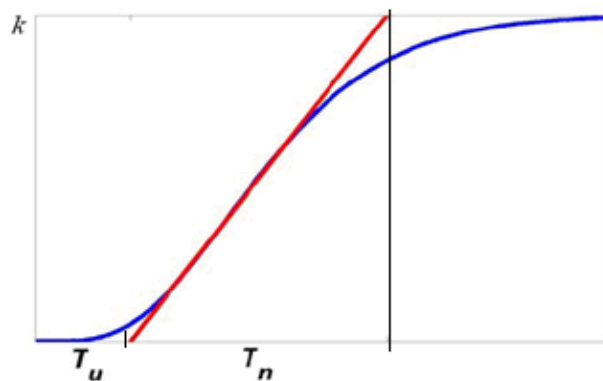
- **Design of I-controller:**

P and D-controllers are disabled. Time constant T_i is slowly decreasing until the undamped oscillations begin in the system. The time constant at this point is **ultimate** T_{ik} .

This value is used to set the optimal integral time constant of I-controller:

- $T_i = 2T_{ik}$ for damped periodic response
- $T_i = 4T_{ik}$ for aperiodic response

When amplification of control loop onto stability limit is impossible. Then we can compute the controller's parameters from parameters of step response of the system: dead time T_u , rise time T_n , and system gain k .



Controller	r_0	T_i	T_d
P	$\frac{T_n}{T_u} \frac{1}{k}$	-	-
PI	$0,9 \frac{T_n}{T_u} \frac{1}{k}$	$3,5 T_u$	-
PID	$1,25 \frac{T_n}{T_u} \frac{1}{k}$	$2 T_u$	$0,5 T_u$

Illustrative example

Design the P-controller for the system given by transfer function using the Ziegler-Nichols method.

$$G_s = \frac{1}{p^3 + 2p^2 + 4p + 1}$$

$$G_R = r_0 = ?$$

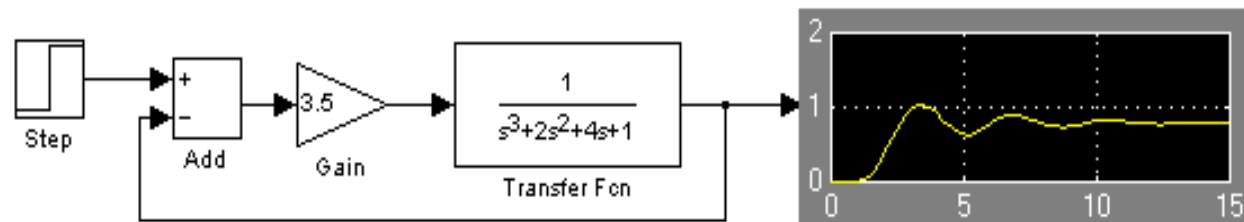
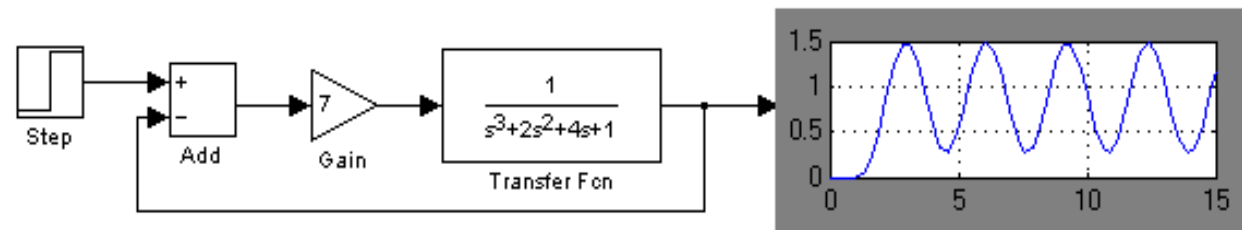
a) Simulation in Matlab:

$$R_{0k} = 7$$

$$T_k = 3$$

b) Optimal setting of P controller: $r_0 = 0,5 \cdot r_{0k} = 0,5 \cdot 7 = 3,5$

Simulation in Matlab:



Ultimate gain r_{0k} using substitution $p = j\omega$

Closed-loop transfer function:

$$G_w = \frac{G_R G_S}{1 + G_R G_S} = \frac{\frac{r_0}{p^3 + 2p^2 + 4p + 1}}{1 + \frac{r_0}{p^3 + 2p^2 + 4p + 1}} = \frac{r_0}{p^3 + 2p^2 + 4p + 1 + r_0}$$

Characteristic polynomial on stability limit: $A(p) = p^3 + 2p^2 + 4p + 1 + r_0 = 0$

After substitution: $A(j\omega) = (j\omega)^3 + 2(j\omega)^2 + 4j\omega + 1 + r_0 = -j\omega^3 - 2\omega^2 + 4j\omega + 1 + r_0 = 0 \Leftrightarrow$

$$\text{Re}\{A(j\omega)\} = 0 \text{ and } \text{Im}\{A(j\omega)\} = 0$$

$$\text{Im:} \quad -\omega^3 + 4\omega = 0 \Rightarrow -\omega(\omega^2 - 4) = 0 \Leftrightarrow \omega_1 = 0 \text{ and } \omega_{1,2} = \pm 2$$

$$\text{Re:} \quad 2\omega^2 + 1 + r_0 = 0 \Rightarrow \text{for } \omega = \pm 2: -8 + 1 + r_0 = 0 \Leftrightarrow r_0 = 7 \quad \text{blue arrow}$$
$$\text{for } \omega = 0: 1 + r_0 = 0 \Leftrightarrow r_0 = -1$$

For $r_0 = 7$ the system exhibits sustained oscillation, therefore the system is on the verge of instability or marginally stable.

Method of standard forms of characteristic polynomial

1. ITAE Criterion $\int_0^{+\infty} t |e(t)| dt$

The time-weighting function penalizes errors that persist for a long period of time. This function helps to derive controller setting where long settling time is permitted.

This is just another empirical method. We are simply using the results of minimization obtained by other people.

Standard forms of characteristic equation depend on the numerator of control transfer function.

For this form of transfer function:

$$a) \quad G_W = \frac{a_0}{a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0}$$

were obtained this forms of characteristic polynomials:

- for the second-order system: $A(p) = p^2 + 1,49 \omega p + \omega^2$
- for third-order system: $A(p) = p^3 + 1,72 \omega p^2 + 2,17 \omega^2 p + \omega^3$
- for forth-order system: $A(p) = p^4 + 2 \omega p^3 + 3,35 \omega^2 p^2 + 2,7 \omega^3 p + \omega^4$

1. ITAE Criterion $\int_0^{+\infty} t |e(t)| dt$

b) For transfer function:
$$G_W = \frac{a_1 p + a_0}{a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0}$$

Characteristic polynomials:

- for the second-order system: $A(p) = p^2 + 10 \omega p + \omega^2$
- for third-order system: $A(p) = p^3 + 2 \omega p^2 + 19,2 \omega^2 p + \omega^3$
- for forth-order system: $A(p) = p^4 + 2,2 \omega p^3 + 4 \omega^2 p^2 + 2,7 \omega^3 p + \omega^4$

c) For transfer function:
$$G_W = \frac{a_2 p^2 + a_1 p + a_0}{a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0}$$

Characteristic polynomials:

- for third-order system: $A(p) = p^3 + 4,7 \omega p^2 + 3,6 \omega^2 p + \omega^3$
- for forth-order system: $A(p) = p^4 + 3 \omega p^3 + 19 \omega^2 p^2 + 9,5 \omega^3 p + \omega^4$
- for fifth-order system: $A(p) = p^5 + 4 \omega p^4 + 12 \omega^2 p^3 + 18 \omega^3 p^2 + 10 \omega^4 p + \omega^5$

Whitteley forms.

The standard forms are derived from the first overshoot amplitude. The settling time is longer than in ITAE, but the overshoot is lower.

a) For transfer function:
$$G_W = \frac{a_0}{a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0}$$

- for second-order system: $A(p) = p^2 + 1,4 \omega p + \omega^2$ 5% overshoot
- for third-order system: $A(p) = p^3 + 2 \omega p^2 + 2 \omega^2 p + \omega^3$ 8%
- For forth-order system: $A(p) = p^4 + 2,6 \omega p^3 + 3,4 \omega^2 p^2 + 2,6 \omega^3 p + \omega^4$ 10%

b) For transfer function:
$$G_W = \frac{a_1 p + a_0}{a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0}$$

- for second-order system: $A(p) = p^2 + 2,5 \omega p + \omega^2$ 10% overshoot
- for third-order system: $A(p) = p^3 + 5,1 \omega p^2 + 6,3 \omega^2 p + \omega^3$ 10%
- for forth-order system: $A(p) = p^4 + 7,2 \omega p^3 + 16 \omega^2 p^2 + 12 \omega^3 p + \omega^4$ 10%

Whitteley forms.

a) For transfer function:
$$G_W = \frac{a_2 p^2 + a_1 p + a_0}{a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + a_0}$$

Characteristic polynomials:

- for second-order system: $A(p) = p^3 + 6,7 \omega p^2 + 6,7 \omega^2 p + \omega^3$ 5% overshoot
- for third-order system: $A(p) = p^4 + 7,9 \omega p^3 + 15 \omega^2 p^2 + 7,9 \omega^3 p + \omega^4$ 8%
- For forth-order system: $A(p) = p^5 + 18 \omega p^4 + 60 \omega^2 p^3 + 69 \omega^3 p^2 + 18 \omega^4 p + \omega^5$ 10%

Multiple roots optimum adjustment

Guarantee the aperiodic response.

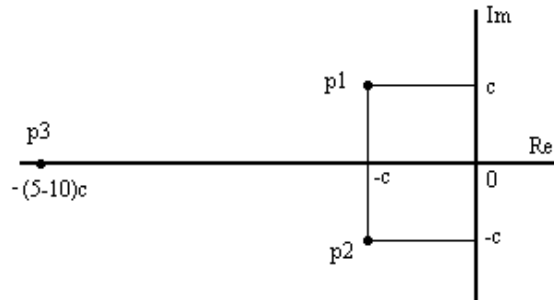
For transfer function: $G_w = \frac{c^n}{(p+c)^n}$

where c is multiple root. Numerators' form is not obligatory, only recommended for optimum adjustment.

- for second-order system: $A(p) = p^2 + 2 p c + c^2$
- for third-order system: $A(p) = p^3 + 3 p^2 c + 3 c^2 p + c^3$
- for forth-order system: $A(p) = p^4 + 4 p^3 c + 6 c^2 p^2 + 4 c^3 p + c^4$

Two dominant roots adjustment

It makes possible to choose two complex conjugate roots and to locate the other roots to the distance of 5-10 multiple of the dominant root.



Standard forms do not depend on numerator in this case.

- for second-order system: $A(p) = [p + (c+jc)] [p + (c-jc)] = p^2 + 2 p c + 2 c^2$
- for third-order system: $A(p) = (p^2 + 2 p c + 2 c^2) [p + kc] = p^3 + p^2 c(k+2) + 2 c^2 (k+1)p + 2kc^3$
- for forth-order system: $A(p) = (p^2 + 2 p c + 2 c^2) (p^2 + 2kcp + k^2c^2) =$
 $p^4 + 2c(k+1)p^3 + c^2 (k^2 +4k +2)p + 2kc^3(2+k)p + 2k^2c^4$

Example solved

Design a PI controller for the system given by the transfer function: $G_s = \frac{1}{3p^2 + 2p}$

$$G_R = r_0 + \frac{r_{-1}}{p}$$

$$G_w = \frac{G_R G_s}{1 + G_R G_s} = \frac{r_0 p + r_{-1}}{3p^3 + 2p^2 + r_0 p + r_{-1}} = \frac{\frac{1}{3}(r_0 p + r_{-1})}{p^3 + \frac{2}{3}p^2 + \frac{r_0}{3}p + \frac{r_{-1}}{3}}$$

a) ITAE

$$A(p) = p^3 + \frac{2}{3}p^2 + \frac{r_0}{3}p + \frac{r_{-1}}{3}$$

Characteristic polynomial:

Standard form of characteristic polynomial: $A(p) = p^3 + 2\omega p^2 + 19,2\omega^2 p + \omega^3$

Compare coefficients:

- at p^2 : $2/3 = 3c \rightarrow c = 2/9$
- at p^1 : $r_0/3 = 3c^2 \rightarrow r_0 = 4/9$
- at p^0 : $r_{-1}/3 = c^3 \rightarrow r_{-1} = 8/243$

$$G_r = 6,4 + \frac{1/9}{p}$$

b) Multiple root adjustment

The characteristic polynomial is:

$$A(p) = p^3 + \frac{2}{3} p^2 + \frac{r_0}{3} p + \frac{r_{-1}}{3}$$

We compare it with the standard form: $A(p) = p^3 + 3 c p^2 + 3 c^2 p + c^3$

- p^2 : $2/3 = 3 c \rightarrow c = 2/9$
- p^1 : $r_0/3 = 3 c^2 \rightarrow r_0 = 4/9$
- p^0 : $r_{-1}/3 = c^3 \rightarrow r_{-1} = 8/243$

$$G_r = \frac{4}{9} + \frac{8/243}{p}$$

c) Dominant roots adjustment

$$A(p) = p^3 + \frac{2}{3} p^2 + \frac{r_0}{3} p + \frac{r_{-1}}{3}$$

The characteristic polynomial is:

We compare it with standard form: $A(p) = p^3 + 7 c p^2 + 12 c^2 p + 10 c^3$

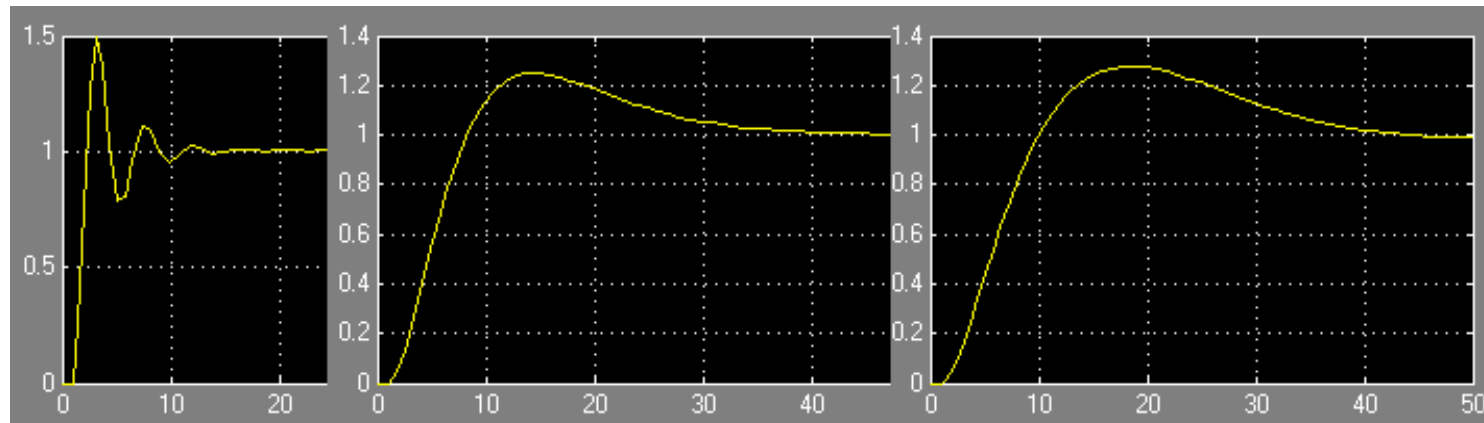
- p^2 : $2/3 = 7 c \rightarrow c = 2/21$
- p^1 : $r_0/3 = 12 c^2 \rightarrow r_0 = 16/49$
- p^0 : $r_{-1}/3 = 10 c^3 \rightarrow r_{-1} = 80/3177$

$$G_r = \frac{16}{49} + \frac{80/3177}{p}$$

Simulation in Matlab

Step responses for:

a) ITAE b) Multiple roots adjustment c) Dominant roots adjustment



Solved example

$$G_S = \frac{1}{p^3 + 3p^2 + 2p} \quad G_R = r_0$$

$$G_w = \frac{G_R G_S}{1 + G_R G_S} = \frac{r_0}{p^3 + 3p^2 + 2p + r_0}$$

- **ITAE**

Characteristic polynomial is: $A(p) = p^3 + 3p^2 + 2p + r_0$

We compare it with standard form: $A(p) = p^3 + 1,72 \omega p^2 + 2,17 \omega^2 p + \omega^3$

Coefficients of the same powers must be equal, but in this case we obtain two equations for ω and the set of equations cannot be solved.

- p^2 : $3 = 1,72 \omega$
- p^1 : $2 = 2,17 \omega^2$
- p^0 : $r_0 = \omega^3$

We cannot even use any other standard form.

Example

Design optimal controller for the system and the input signal given below.

$$G_s = \frac{1}{8p^2 + 22p + 12} \quad w(t) = k \cdot t$$

References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.

ISBN: 81-7409-107-6

[2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing

[3] Matlab&Simulink: R2015a

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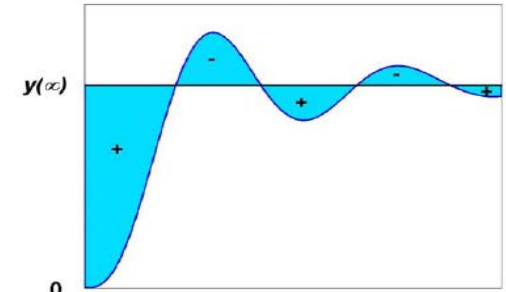
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12. Standard Methods of Control Adjustment

Quadratic integral criterion - integral of the square error

Object function:
$$I_q = \int_0^{+\infty} e^2(t) dt = \int_0^{+\infty} [y(\infty) - y(t)]^2 dt = \int_0^{+\infty} \overline{y(t)} dt \rightarrow \min$$



This process consists of several steps:

1. Create control transfer function $G_w(p)$, or transfer function of disturbance $G_z(p)$.
2. Calculate output function $Y(p)$.
3. Calculate steady state value of output: $y(\infty) = \lim_{t \rightarrow \infty} y(t) = \lim_{p \rightarrow 0} pY(p) = konst$
4.
$$\bar{Y}(p) = Y(p) - \frac{1}{p} y(\infty) = \frac{b_{n-1}p^{n-1} + b_{n-2}p^{n-2} + \Lambda + b_1p + b_0}{a_n p^n + a_{n-1}p^{n-1} + \Lambda + a_1p + a_0}$$
5. Create Hurwitz matrix H_n .
6. Create modified Hurwitz matrix H_1 .
7. The quadratic control area is computed from the following formula:
$$I_q = \frac{1}{2a_n} \frac{\det(H_1)}{\det(H_n)}$$
8. For **min** I_q determine these conditions:
$$\frac{\partial I_q}{\partial r_0} = 0 \quad \frac{\partial I_q}{\partial r_{-1}} = 0 \quad \frac{\partial I_q}{\partial r_1} = 0$$
9. At the end check the stability.

Example

Determine I-controller using quadratic integral criterion:

$$G_S(p) = \frac{1}{2p^2 + 6p + 4} \quad G_R = \frac{r_{-1}}{p} \quad z(t) = 1$$

$$1. \quad \text{Disturbance transfer function: } G_z = \frac{G_S(p)}{1 + G_R(p)G_S(p)} = \frac{\frac{1}{2p^2 + 6p + 4}}{1 + \frac{r_{-1}}{2p^2 + 6p + 4}} = \frac{p}{2p^3 + 6p^2 + 4p + r_{-1}}$$

$$2. \quad \text{Output signal: } Y(p) = G_z(p) \cdot Z(p) = \frac{p}{2p^3 + 6p^2 + 4p + r_{-1}} \cdot \frac{1}{p} = \frac{1}{2p^3 + 6p^2 + 4p + r_{-1}}$$

$$3. \quad \text{Steady state value of output: } y(\infty) = \lim_{p \rightarrow 0} p \cdot Z(p) = p \cdot \frac{1}{2p^3 + 6p^2 + 4p + r_{-1}} = 0$$

$$\bar{Y}(p) = Y(p) - \frac{1}{p} y(\infty) = \frac{1}{2p^3 + 6p^2 + 4p + r_{-1}}$$

4. Hurwitz determinant:

$$H_n = \begin{bmatrix} 6 & r_{-1} & 0 \\ 2 & 4 & 0 \\ 0 & 6 & r_{-1} \end{bmatrix} = r_{-1}(24 - 2r_{-1}) = 24r_{-1} - 2r_{-1}^2$$

5. Modified Hurwitz determinant: $H_1 = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 4 & 0 \\ 0 & 6 & r_{-1} \end{bmatrix} = 1 \cdot 12 = 12$

$$h_{11} = (-1)^0 b_{n-1}^2 = 0$$

$$h_{12} = (-1)^1 (b_{n-2}^2 - 2b_{n-1}b_{n-3}) = 0$$

$$h_{13} = (-1)^2 (b_{n-3}^2) = 1$$

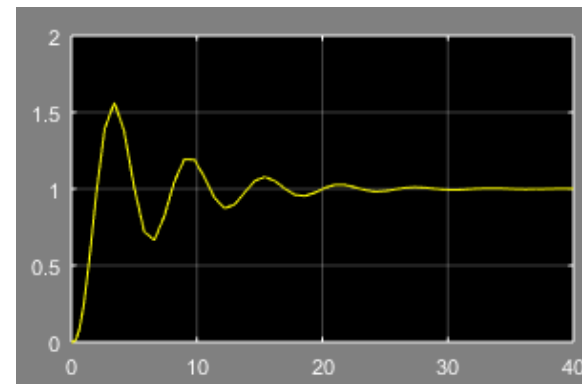
6. Quadratic control area: $I_q = \frac{1}{2a_n} \frac{\det(H_1)}{\det(H_n)} = \frac{1}{4} \cdot \frac{12}{24r_{-1} - 2r_{-1}^2}$

7. Minimization: $\frac{\partial I_q}{\partial r_{-1}} = \frac{-3(24 - 4r_{-1})}{(24r_{-1} - 2r_{-1}^2)^2} = 0 \leftrightarrow (24 - 4r_{-1}) = 0 \quad r_{-1} = 6$

8. Optimal setting of I-controller:

$$G_R = \frac{6}{p}$$

9. Simulation of the step response in Matlab:



Problem

Determine I-controller using quadratic integral criterion:

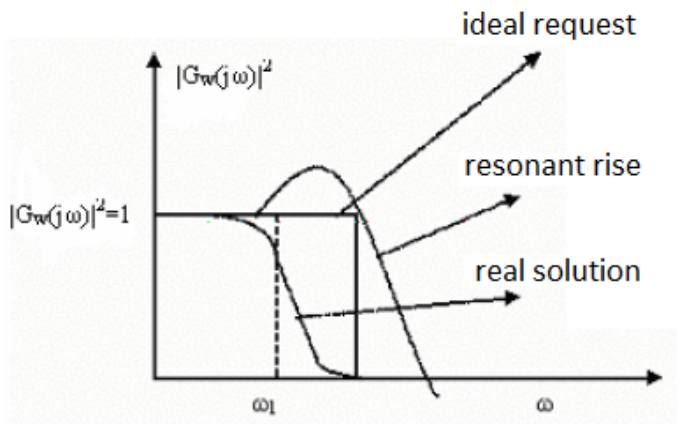
$$G_S = \frac{1}{2,5p^3 + 8p^2 + 6,5p + 1} \quad G_R = \frac{r-1}{p} \quad w(t) = 1$$

Solution: $r_{-1} = -0,6529$ (unstable solution)

$r_{-1} = 0,2432$ (optimal setting)

Optimal Module Criterion

The target of control is described by this **objective function**: $|G_w(j\omega)| \rightarrow 1$



$$|G_w(j\omega)| = 1 \Rightarrow |G_w(j\omega)|^2 = 1$$

$$|G_w(j\omega)| = |Y(j\omega)|$$

$$|G_w(j\omega)|^2 = |Y(j\omega)|^2 = Y(j\omega) \cdot Y(-j\omega) = \frac{B_m \omega^{2m} + B_{m-1} \omega^{2(m-1)} + \dots + B_1 \omega^2 + B_0}{A_n \omega^{2n} + A_{n-1} \omega^{2(n-1)} + \dots + A_1 \omega^2 + A_0}$$

Where:

$$A_0 = a_0^2$$

$$A_1 = a_1^2 - 2a_0 a_2$$

$$A_2 = a_2^2 - 2a_1 a_3 + 2a_0 a_4$$

$$A_{n-1} = a_{n-1}^2 - 2a_{n-2} \cdot a_n$$

$$A_n = a_n^2$$

$$B_0 = b_0^2$$

$$B_1 = b_1^2 - 2b_0 b_2$$

$$B_2 = b_2^2 - 2b_1 b_3 + 2b_0 b_4$$

$$B_{n-1} = b_{n-1}^2 - 2b_{n-2} \cdot b_n$$

$$B_n = b_n^2$$

Optimal Module Criterion

To reach $|G_w(j\omega)|^2 = 1$ must the coefficients: $A_0 = B_0$, $A_1 = B_1$, , $A_n = B_n$

In fact we are able to achieve:

$$\frac{B_0}{A_0} \geq \frac{B_1}{A_1} \geq \frac{B_2}{A_2} \geq \dots \geq \frac{B_i}{A_i}$$

From this set of conditions for monotone decreasing function we can obtain the parameters of regulator.

Illustrative example: Determine I-controller using Optimal Module Criterion.

$$G_S(p) = \frac{1}{2p^2 + 6p + 4} \quad G_R = \frac{r-1}{p} \quad z(t) = 1$$

$$1. \quad G_Z = \frac{G_S(p)}{1 + G_R(p)G_S(p)} = \frac{\frac{1}{2p^2 + 6p + 4}}{1 + \frac{r-1}{2p^2 + 6p + 4}} = \frac{p}{2p^3 + 6p^2 + 4p + r - 1}$$

$$2. \quad Y(p) = G_z(p) \cdot Z(p) = \frac{p}{2p^3+6p^2+4p+r_{-1}} \cdot \frac{1}{p} = \frac{1}{2p^3+6p^2+4p+r_{-1}}$$

$$3. \quad y(\infty) = \lim_{p \rightarrow 0} p \cdot Z(p) = p \cdot \frac{1}{2p^3+6p^2+4p+r_{-1}} = 0$$

4. Computation of coefficients A_0, A_1, \dots and B_0, B_1, \dots of $Y(j\omega)^2$ module.

$$\begin{aligned} B_0 &= b_0^2 = 1 & A_0 &= a_0^2 = r_{-1}^2 \\ B_1 &= 0 & A_1 &= a_1^2 - 2a_2a_0 = 16 - 2 \cdot 6 \cdot r_{-1} \end{aligned}$$

5. Conditions for monotone decreasing function:

$$\frac{B_0}{A_0} \geq \frac{B_1}{A_1}$$

$$B_0A_1 = B_1A_0$$

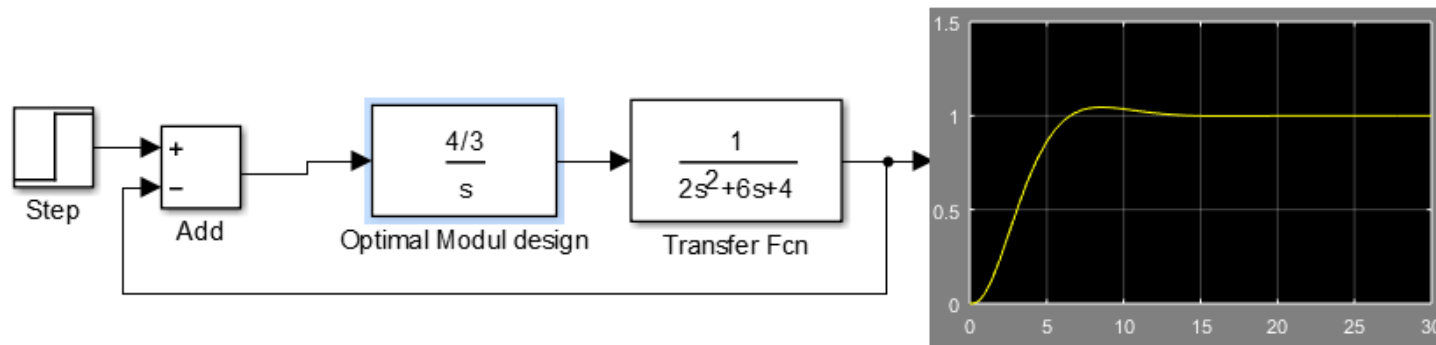
$$1 \cdot (16 - 12r_{-1}) = 0$$

$$r_{-1} = \frac{16}{12} = \frac{4}{3}$$

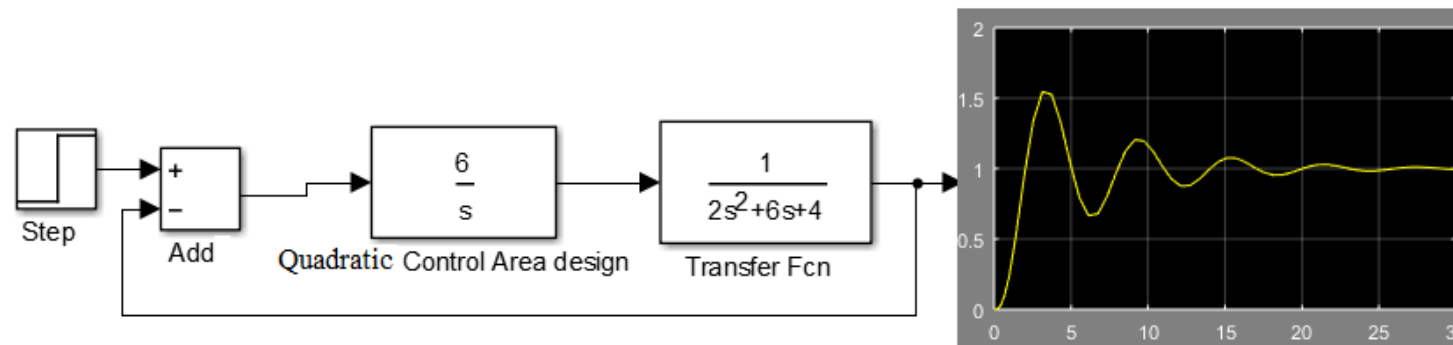
6. Optimal setting of I-controller: $G_R = \frac{4/3}{p}$

Simulation of the step response in Matlab

- Optimum Module design:



- Comparison with Quadratic Control Area design



Control design in state space

- **Controllability**

A linear system $\dot{x}(t) = A x(t) + B u(t)$

$$y(t) = C x(t) + D u(t)$$

is said to be completely state controllable if there exist an input $u(t)$ that can drive the system from any given initial state $x_0(t_0 = 0)$ to any other desired state $x(t)$.

The system is completely controllable if and only if the controllability matrix $Q_c = [B \ AB \ A^2B \ \dots \ A^{n-1}B]$ is of rank n .

- **Observability**

The linear system is completely observable if every initial state $x(0)$ can be determined from output $y(t)$ over a finite time interval.

The system is completely observable if and only if the observability matrix is of rank n . $Q_0 = \begin{bmatrix} C \\ CA \\ \vdots \\ C A^{n-1} \end{bmatrix}$

Illustrative example

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = [1 \ 0 \ 0]$$

Compute the controllability and observability matrices and decide if the model is controllable and observable.

$$Q_c = [B \ AB \ A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{bmatrix} \quad \text{rank}(Q_c) = 3$$

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{rank}(Q_o) = 3$$

System is controllable and observable.

To compute the controllability and observability matrix we can use the Matlab functions: **ctrb**, **obsv**

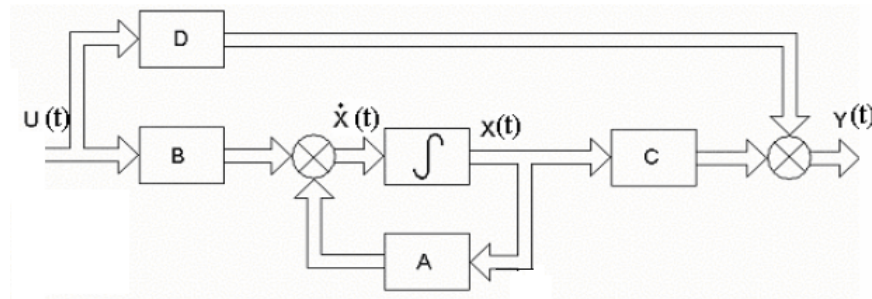
To compute the rank of matrix we can use the Matlab function: **rank**

Poleplacement design

Control system is given by

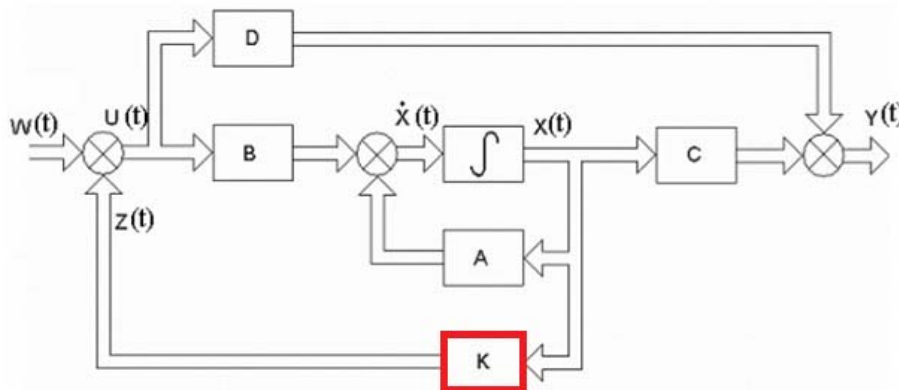
$$\dot{x}(t) = A x(t) + B u(t)$$

$$y(t) = C x(t) + D u(t)$$



Feedback gain matrix K:

The eigen values of the system matrix A are roots of the characteristic polynomial. On application of the state variable feedback, the eigen values of the matrix A can be adjusted to a desired value.



Following equations are obtained:

$$u(t) = w(t) - K x(t)$$

$$\dot{x}(t) = A x(t) + B(w(t) - K x(t))$$

$$\dot{x}(t) = (A - B \cdot K)x(t) + B u(t)$$

New matrix of dynamics:

$$\tilde{A} = (A - B \cdot K)$$

Illustrative example

$$A = \begin{bmatrix} -0,05 & 0,05 & 0 \\ 0,05 & -0,1 & 0,05 \\ 0 & 0,05 & -0,1 \end{bmatrix} \quad B = \begin{bmatrix} 0,5 \\ 0 \\ 0 \end{bmatrix}$$

The original roots are located: $\text{eig}(A)$:
 $p_1 = -0,1623$
 $p_2 = -0,0777$
 $p_3 = -0,0099$

The closed-loop poles are to be placed at:
 $p_1 = -0,2 + 0,2j$
 $p_2 = -0,2 - 0,2j$
 $p_3 = -1$

Determine the feedback gain matrix K.

Solution:

The desired characteristic polynomial:

$$\begin{aligned} A(p) &= (p - p_1) \cdot (p - p_2) \cdot (p - p_3) = (p + 0,2 - 0,2j) \cdot (p + 0,2 + 0,2j) \cdot (p + 1) = \\ &= p^3 + 1,4p^2 + 0,48p + 0,018 \end{aligned}$$

$$\begin{aligned} A - Bk &= \begin{bmatrix} -0.05 & 0.05 & 0 \\ 0.05 & -0.1 & 0.05 \\ 0 & 0.05 & -0.1 \end{bmatrix} - \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} = \begin{bmatrix} -0.05 & 0.05 & 0 \\ 0.05 & -0.1 & 0.05 \\ 0 & 0.05 & -0.1 \end{bmatrix} - \begin{bmatrix} 0.5k_1 & 0.5k_2 & 0.5k_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \\ &= \begin{bmatrix} -0.05 - 0.5k_1 & 0.05 - 0.5k_2 & -0.5k_3 \\ 0.05 & -0.1 & 0.05 \\ 0 & 0.05 & -0.1 \end{bmatrix} \end{aligned}$$

Characteristic polynomial: $\det(pI - A - B \cdot k) = p^3 + (0,5k_1 + 0,25)p^2 + (0,025k_2 + 0,1k_1 + 0,015)p + 0,00125k_3 + 0,0025k_2 + 0,00375k_1 + 0,000125$

Comparing the coefficients of the same powers of p we obtain the state feedback gains in matrix K .

$$\begin{aligned}0,5k_1 + 0,25 &= 1,4 \\0,025k_2 + 0,1k_1 + 0,015 &= 0,48 \\0,00125k_3 + 0,0025k_2 + 0,00375k_1 + 0,000125 &= 0,08\end{aligned}$$

$$\begin{aligned}k_1 &= 2,3 \\k_2 &= 9,4 \\k_3 &= 38,2\end{aligned}$$

$$K = [2,3 \quad 9,4 \quad 38,2]$$

Poleplacement design we can solve using Ackermann's formula implemented in Matlab.

```
» A=[-0.05 0.05 0;0.05 -0.1 0.05;0 0.05 -0.1];  
» B=[0.5; 0; 0];  
» P=[-0.2+0.2i;-0.2-0.2i;-1]; // where P is a vector of desired poles  
» K=ACKER(A,B,P)
```

K =

2.3000 9.4000 38.2000

Pole-placement according to the time of the first overshoot.

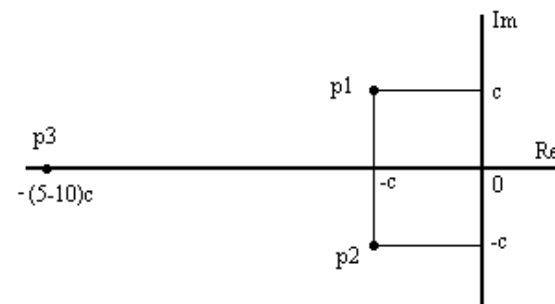
The poles location affects the time of the first overshoot. The corresponding dominant complex conjugate roots can be computed from this empirical formula:

$$T_p \cong \frac{3}{c} \rightarrow c \cong \frac{3}{T_p}$$

Where c corresponds to two complex conjugated poles:

$$\begin{aligned} p_1 &= -c + cj \\ p_2 &= -c - cj \\ p_3 &= -k \cdot c \end{aligned}$$

The other roots have to be chosen further away of the imaginary axis.



Linear-quadratic regulator (LQR) design

The theory of optimal control is concerned with operating a dynamic system at minimum cost.

The system dynamics are described by a set of linear differential equations and the cost is described by a quadratic function is called the LQ problem - **linear-quadratic regulator (LQR)**.

The settings of a controller are found by using a mathematical algorithm that minimizes a cost function with weighting factors supplied by an engineer.

The quadratic cost function:

$$J = \frac{1}{2} \int_0^{\infty} (x^T Q x + u^T R u) dt$$

Where: Q, R are weighting matrixes

x is a vector of state variable

u is a vector of inputs

The feedback control law that minimizes the value of the cost is: $u(t) = -K x(t)$

Where K is given by: $K = -R^{-1} B^T P$

Riccati equation

Matrix P is found by solving the continuous time algebraic Riccati equation:

$$P A + A^T P - P B R^{-1} B^T P + Q = 0$$

Solution can be found in Matlab:

$$[K, P, L] = \text{LQR} (A, B, Q, R)$$

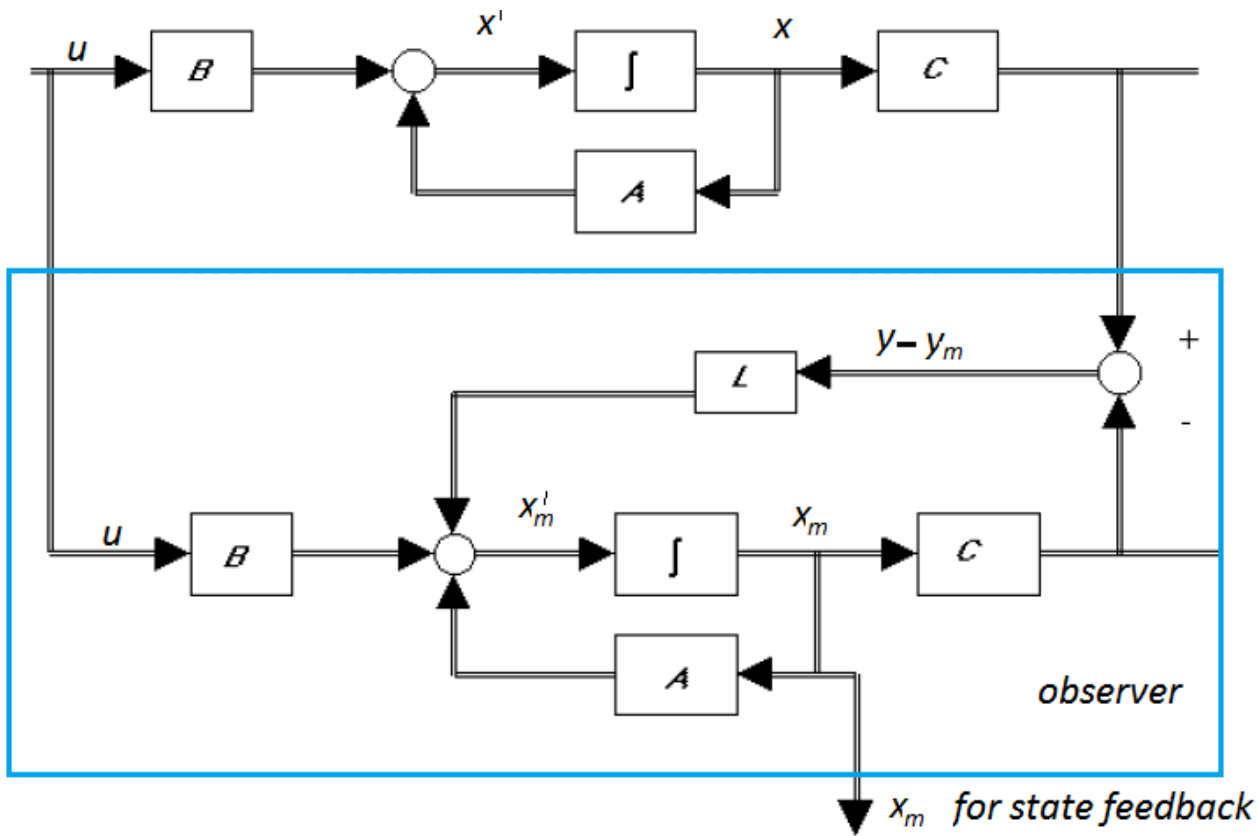
where: K is a vector of the feedback gain

P is Riccati matrix - solution of Riccati equation

L is a vector of eigen values of the system: $L = \text{eig}(A-BK)$

Observer

Observer is a dynamic subsystem of a control system that calculates those state variables that can not be directly measured on an object. If a system is observable, it is possible to fully reconstruct the system state from its output measurements using the state observer.



$$x' = Ax + Bu$$

$$y = Cx$$

$$x'_m = Ax_m + Bu + L(y - y_m)$$

$$y_m = Cx_m$$

$$E(t) = x(t) - x_m(t)$$

$$\lim_{t \rightarrow \infty} E(t) = 0$$

$$\begin{aligned} E'(t) &= x'(t) - x'_m(t) \\ &= Ax + Bu - Ax_m - Bu - L(Cx - Cx_m) \end{aligned}$$

$$E'(t) = (A - LC)E(t)$$

Typical observer model

The output of the observer may be subtracted from the output of the plant and then multiplied by a matrix L . This is then added to the equations for the state of the observer to produce a so-called *Luenberger observer*, defined by the equations below. The variables of a state observer are commonly denoted by a x_m or y_m to distinguish them from the variables of the equations satisfied by the physical system.

The observer is called asymptotically stable if the observer error $E(t) = x(t) - x_m(t)$ converges to zero.

The Luenberger observer is therefore asymptotically stable when the matrix $(A-LC)$ has all the eigenvalues with negative real parts.

For control purposes the output of the observer system is fed back to the input of both the observer and the plant through the gains matrix K .

$$u(t) = -Kx_m(t)$$

We can choose K and L independently without harm to the stability of the systems. The poles of the observer $(A-LC)$ are usually chosen to converge 3-10 times faster than the poles of the system $(A-BK)$.

Illustrative example

Assume system:

$$A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \quad C = [1 \quad 0]$$

$$\det(pI - A) = \det \begin{bmatrix} p+3 & 1 \\ -2 & p \end{bmatrix} = p^2 + 3p + 2 = (p+1)(p+2) \quad p_1 = -1 \quad p_2 = -2$$

Eigenvalues of $(A-LC)$ choice: -6; -6

Desired characteristic polynomial: $(p+6)^2 = p^2 + 12p + 36$

Characteristic polynomial of Luenberger observer:

$$\tilde{A} = A - LC = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} [1 \quad 0] = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} l_1 & 0 \\ l_2 & 0 \end{bmatrix} = \begin{bmatrix} -3-l_1 & -1 \\ 2-l_2 & 0 \end{bmatrix}$$

$$\det(pI - \tilde{A}) = \det \begin{bmatrix} p+3+l_1 & 1 \\ -2+l_2 & p \end{bmatrix} = p^2 + 3p + l_1p + 2 - l_2 = p^2 + p(3+l_1) + 2-l_2$$

Comparing the both polynomials:

$$3 + l_1 = 12 \rightarrow l_1 = 9$$

$$2 + l_2 = 36 \rightarrow l_2 = -34$$

$$L = \begin{bmatrix} 9 \\ -34 \end{bmatrix}$$

Illustrative example in Matlab

You can use *place* for estimator gain: $L = \text{place}(A', C', P)$

The length of P must match the row size of A .

```
>> A=[-3 -1;2 0];
```

```
>> C=[1 0];
```

```
>> P=[-5.9 -6.1];           //The length of  $P$  must match the row size of  $A$  and be placed separately
```

```
>> L=place(A',C',P)'
```

L =

9.0000

-33.9900

References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.

ISBN: 81-7409-107-6

[2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing

[3] Matlab&Simulink: R2015a

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MINISTRY OF EDUCATION,
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13. Discrete Time Control System

Analysis of sampled-data control system

The analysis of sampled-data control system is carried out in Z-domain using Z-transform.

The chapter gives explanation of:

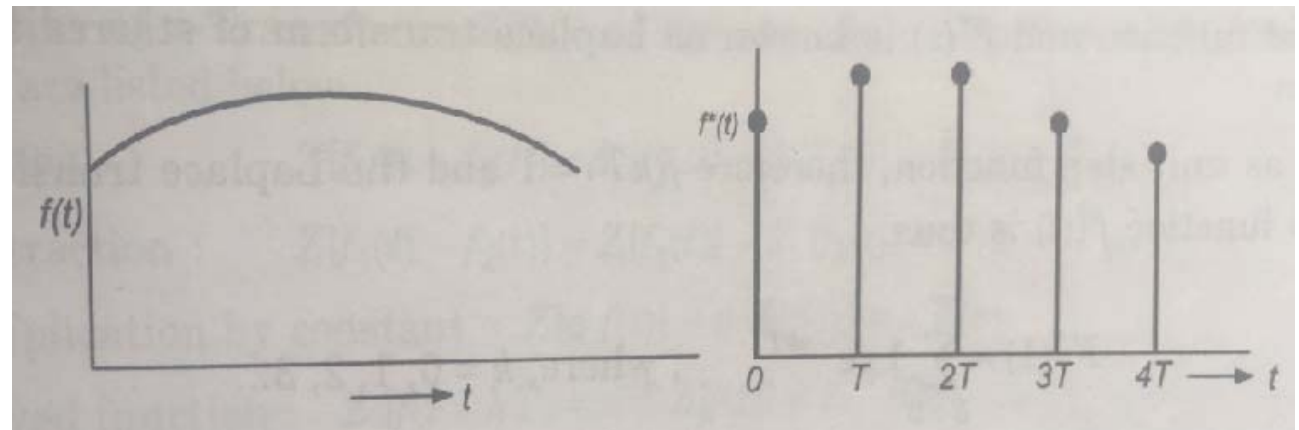
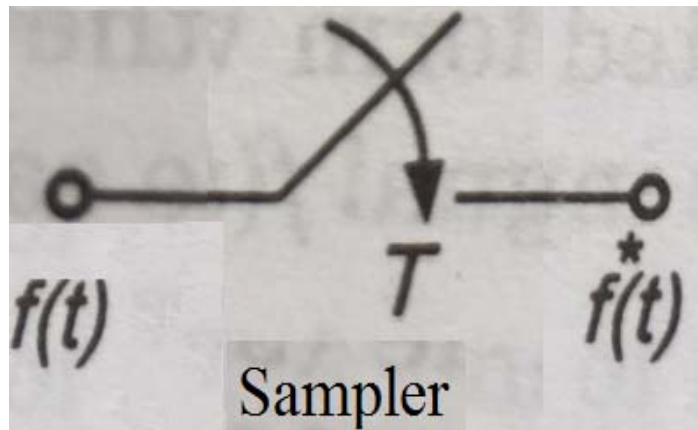
- Sampling process
- Z-transform
- Z-transform of some useful function
- Inverse Z-transform
- Reconstruction of signal – minimum sampling frequency
- Difference equations and Z-transfer function
- Block diagram of transfer function
- Stability analysis of sampled-data control system
- Discrete impulse response
- Discrete step response

Sampling process

The sampler converts a continuous signal into a sequence of pulses, wherein the magnitude of the pulse gives the value of the input signal at the instant of sampling.

The pulse considered is of very short time duration and thus approximated as an impulse.

The sampled output $f(kT)$ of ideal sampler has a time period T .



Continuous signal

Sampled signal

Z-transform of discrete function $f(kT)$

Definition formula:

$$F(z) = Z\{f(kT)\} = \sum_{k=0}^{\infty} f(kT) \cdot z^{-k}$$

In series form $F(z)$ can be written: $F(z) = f(0)z^0 + f(T)z^{-1} + f(2T)z^{-2} + f(3T)z^{-3} + \dots + f(kT)z^{-k}$

The value $f(t)$ at sampling instants kT are given by the coefficients z^{-k} in term above.

$f(0), f(T), f(2T), \dots$ are the values of function $f(t)$ at instants $k = 1, 2, 3, \dots$ respectively.

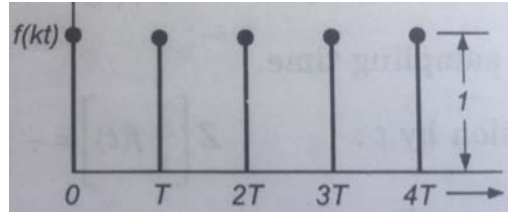
Properties of Z-transform

1. Addition: $Z\{f_1(t) + f_2(t)\} = F_1(z) + F_2(z)$
2. Subtraction: $Z\{f_1(t) - f_2(t)\} = F_1(z) - F_2(z)$
3. Multiplication by constant: $Z\{a f(t)\} = a F(z)$
4. Delayed function: $Z\{f(t - kT)\} = z^{-k} F(z)$
5. Multiplication by T : $Z\{t f(t)\} = -T z (dF(z)/dz)$
6. Summation:
$$\sum_{k=0}^{\infty} Z\{f(kT)\} = \frac{z}{z-1} F(z)$$
7. Initial value theorem: $f(0) = \lim_{z \rightarrow \infty} F(z)$
8. Final value theorem: $f(\infty) = \lim_{z \rightarrow 1} (z-1)F(z) = \lim_{z \rightarrow 1} (1-z^{-1})F(z)$

Z-transform of some useful function

1. Discrete unit step function:

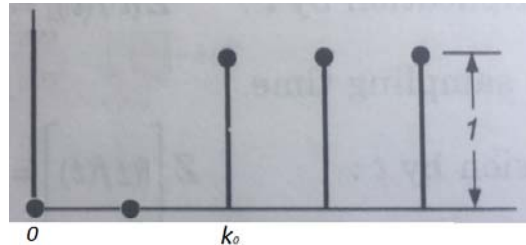
$$f(kT) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases}$$



$$F(z) = \sum_{k=0}^{\infty} f(kT)z^{-k} = 1z^0 + 1z^{-1} + 1z^{-2} + \dots = 1 + z^{-1} + z^{-2} + \dots = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

2. Delayed unit step function:

$$f((k - k_0)T) = \begin{cases} 1 & \text{for } k \geq k_0 \\ 0 & \text{for } k < k_0 \end{cases}$$



$$F(z) = \sum_{k=0}^{\infty} f((k - k_0)T)z^{-k} = 1z^{k_0} + 1z^{-(k_0+1)} + 1z^{-(k_0+2)} + \dots = z^{-k_0} (1 + z^{-1} + z^{-2} + \dots) = \frac{z^{-k_0}}{1 - z^{-1}}$$

3. Discrete unit impulse function:

$$\delta(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

$$F(z) = Z\{\delta(k)\} = 1$$

4. Delayed unit impulse function:

$$\delta(k - k_0) = \begin{cases} 1 & \text{for } k = k_0 \\ 0 & \text{for } k \neq k_0 \end{cases}$$

$$F(z) = Z\{\delta(k - k_0)\} = 1 \cdot z^{k_0}$$

Z-transform of some useful function

4. Exponential function:

$$f(kT) = e^{-akT}$$

$$F(z) = \sum_{k=0}^{\infty} f(kT)z^{-k} = \sum_{k=0}^{\infty} e^{-akT}z^{-k} = 1 + e^{-aT}z^{-1} + e^{-a2T}z^{-2} + e^{-a3T}z^{-3} \dots = \frac{1}{1 - e^{-aT}z^{-1}} = \frac{z}{z - e^{-aT}}$$

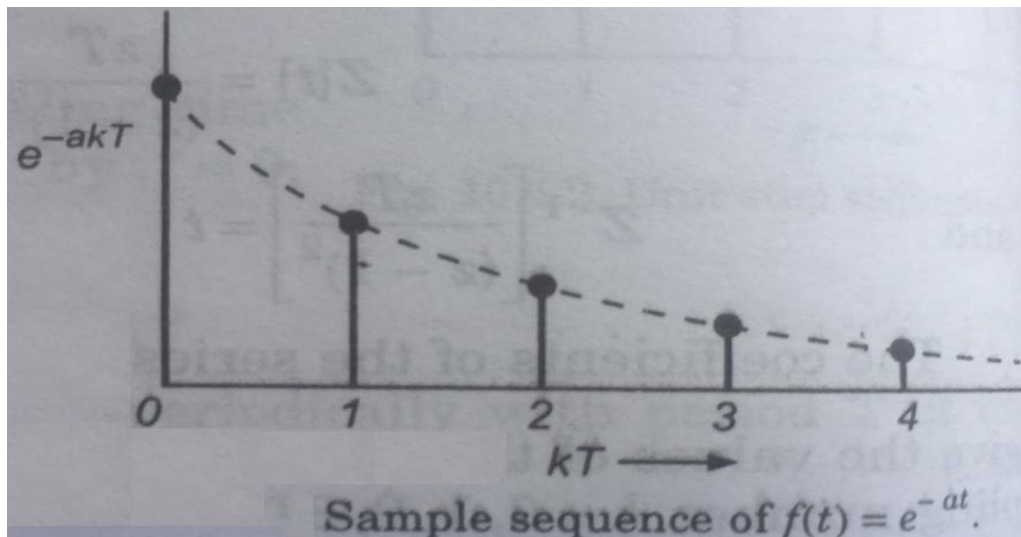


Table of Z -transform pairs

$F(p)$	$f(t)$	$F(z)$
$\frac{1}{p}$	$u_1(t)$	$\frac{z}{z-1}$
$\frac{1}{p^2}$	t	$\frac{Tz}{(z-1)^2}$
$\frac{1}{p^3}$	$\frac{1}{2}t^2$	$\frac{T^2 z(z+1)}{2(z-1)^3}$
$\frac{1}{p^{k+1}}$	$\frac{1}{k!}t^k$	$\lim_{a \rightarrow 0} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial a^k} \left(\frac{z}{z - e^{-aT}} \right)$
$\frac{1}{p+a}$	e^{-at}	$\frac{z}{z - e^{-aT}}$
$\frac{1}{(p+a)^2}$	$t \cdot e^{-at}$	$\frac{Tze^{-aT}}{(z - e^{-aT})^2}$
$\frac{a}{p(p+a)}$	$1 - e^{-at}$	$\frac{z(1 - e^{-aT})}{(z-1)(z - e^{-aT})}$
$\frac{a}{p^2(p+a)}$	$t - \frac{1 - e^{-at}}{a}$	$\frac{Tz}{(z-1)^2} - \frac{z(1 - e^{-aT})}{a(z-1)(z - e^{-aT})}$
$\frac{\omega_0}{p^2 + \omega_0^2}$	$\sin \omega_0 t$	$\frac{z \sin \omega_0 T}{z^2 - 2z \cos \omega_0 T + 1}$
$\frac{p}{p^2 + \omega_0^2}$	$\cos \omega_0 t$	$\frac{z^2 - z \cos \omega_0 T}{z^2 - 2z \cos \omega_0 T + 1}$
$\frac{b-a}{(p+a)(p+b)}$	$e^{-at} - e^{-bt}$	$\frac{z}{z - e^{-aT}} - \frac{z}{z - e^{-bT}}$

F(p)	f(t)	F(z)
$\frac{(b-a)(p+c)}{(p+a)(p+b)}$	$(c-a)e^{-at} + (b-c)e^{-bt}$	$\frac{z(c-a)}{z-e^{-aT}} + \frac{z(b-c)}{z-e^{-bT}}$
$\frac{ab}{p(p+a)(p+b)}$	$1 + \frac{b}{a-b}e^{-at} - \frac{a}{a-b}e^{-bt}$	$\frac{z}{z-1} + \frac{bz}{(a-b)(z-e^{-aT})} - \frac{az}{(a-b)(z-e^{-bT})}$
$\frac{a^2}{p(p+a)^2}$	$1 - (1+at)e^{-at}$	$\frac{z}{z-1} - \frac{z}{z-e^{-aT}} - \frac{aTze^{-aT}}{(z-e^{-aT})^2}$
$\frac{a^2}{p^2(p+a)^2}$	$at - 2 + (at+2)e^{-at}$	$\frac{1}{a} \left[\frac{(aT+2)z - 2z^2}{(z-1)^2} + \frac{2z}{z-e^{-aT}} + \frac{aTze^{-aT}}{(z-e^{-aT})^2} \right]$
$\frac{a^2b}{p(p+b)(p+a)^2}$	$\frac{2ab-b^2}{(a-b)^2}e^{-at} - \frac{a^2}{(a-b)^2}e^{-bt} + \frac{ab}{a-b}te^{-at} + 1$	$\frac{z[2ab-b^2]}{(a-b)^2(z-e^{-aT})} - \frac{a^2z}{(a-b)^2(z-e^{-bT})} + \frac{abTze^{-aT}}{(a-b)(z-e^{-aT})^2} + \frac{z}{z-1}$
$\frac{a^2b^2}{p^2(p+a)(p+b)}$	$\frac{a^2}{a-b}e^{-bt} - (a+b) - \frac{b^2}{a-b}e^{-at} + abt$	$\frac{a^2z}{(a-b)(z-e^{-bT})} - \frac{z(a+b)}{z-1} - \frac{b^2z}{(a-b)(z-e^{-aT})} + \frac{abTz}{(z-1)^2}$
$\frac{a^2b^2(p+c)}{p^2(p+a)(p+b)}$	$abct + [ab - c(a+b)] - \frac{b^2(c-a)}{a-b}e^{-at} - \frac{a^2(b-c)}{a-b}e^{-bt}$	$\frac{abcTz}{(z-1)^2} + \frac{ab-c(a+b)z}{z-1} - \frac{b^2z(c-a)}{(a-b)(z-e^{-aT})} - \frac{a^2z(b-c)}{(a-b)(z-e^{-bT})}$

Inverse Z-transform

Inverse Z-transform is used to obtain function $f(k)$ from the corresponding Z-transform $F(z)$. The inverse Z-transform gives the value of the continuous time function $f(t)$ at sampling instants only – at kT . It does not explain the behavior of continuous time between consecutive sampling instants.

The inverse Z-transform of rational function is obtained by partial fraction and power series methods.

Partial fraction expansion method

Illustrative example:

$$F(z) = \frac{z + 1}{z^2 - 1,5z + 0,5} = \frac{4}{z - 1} + \frac{-3}{z - 0,5}$$

$$f(kT) = 4 \cdot (1)^{k-1} - 3 \cdot (0,5)^{k-1}$$

$$f(0) = 0$$

$$f(1) = 1$$

$$f(2) = 2,5$$

$$f(3) = 3,25$$

Inverse Z-transform

Power series method

Using long division as follows:

$$F(z) = \frac{2 - z^{-1}}{1 - 2z^{-1} + z^{-2}}$$

$$F(z) = (2 - z^{-1}) : (1 - 2z^{-1} + z^{-2}) = 2 + 3z^{-1} + 4z^{-2} + \dots$$

The first few terms of $f(kT)$ at the sampling instants $k = 0, 1, 2, 3 \dots$ are given by the $F(z)$ series:

$$f(0) = 2$$

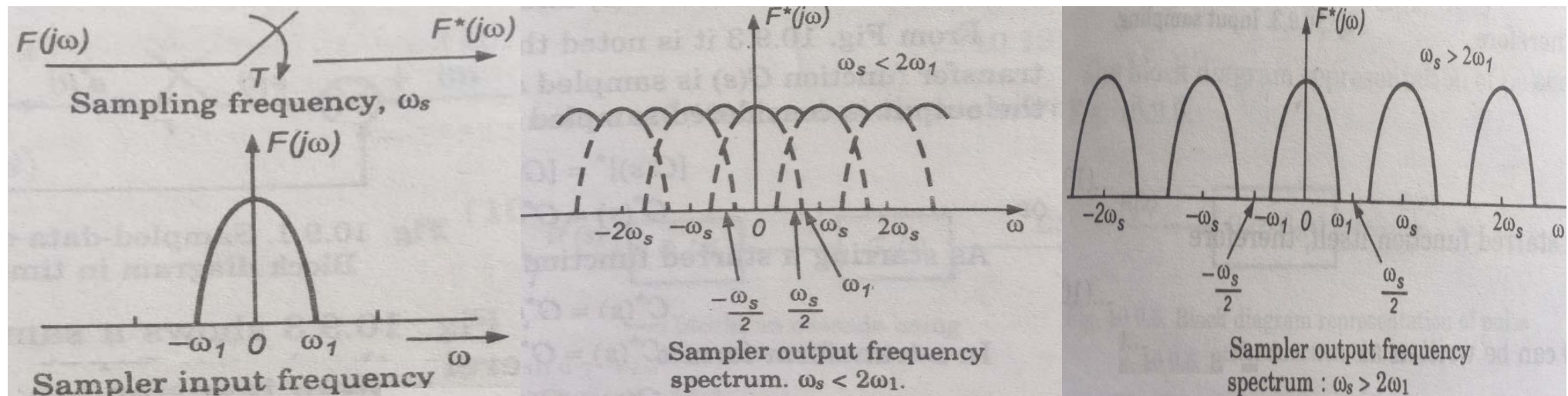
$$f(1) = 3$$

$$f(2) = 4$$

The power series method required long term division to get more terms, using partial fraction decomposition appears to be more convenient.

Reconstruction of signal – minimum sampling frequency

The sampler output contains original components of the input frequency ω_1 and in addition complementary high frequency components.



Shannon-Kotelnikov theorem:

$$\omega_s = \frac{2\pi}{T} < 2\omega_1$$

Complementary frequency components overlap each other. The original signal is not reconstructed.

$$\omega_s = \frac{2\pi}{T} \geq 2\omega_1$$

The original frequency component is isolated from complementary components, the original signal is reconstructed.

Differential equation and Z-transfer function

The analysis of sampled-data control system can be carried out in terms of difference equation.

The time function at the sampling instants can be written as follows:

$$y(k-2) + 3y(k-1) + 2y(k) = 2u(k-2) + 3u(k-1) + u(k)$$

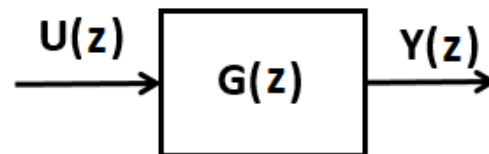
The Z-transform is related to difference equation in a similar manner as the Laplace transform to differential equation. Z-transform converts difference equation with constant coefficients into algebraic equation in terms of „z“. The initial data is already included.

$$\begin{aligned} z^{-2}Y(z) - z^{-1} \cdot y(0) - z^{-2}y(1) + 3z^{-1}Y(z) - 3z^{-1}y(0) + 2Y(z) = \\ = 2z^{-2} - 2z^{-1} \cdot u(0) - 2z^{-2}u(1) + 3z^{-1}U(z) - 3z^{-1}u(0) + U(z) \end{aligned}$$

$y(k)$, $Y(z)$ is considered to be the output of a control system

$u(k)$, $U(z)$ is considered to be the input of a control system

The transfer function $G(z)$ of a control system is defined as the ratio of the Z-transform of the output variable $Y(z)$ to the Z-transform of the input variable $U(z)$ assuming all initial conditions as zero.



Z-transfer function

$$Y(z)(z^{-2} + 3z^{-1} + 2) = U(z)(2z^{-2} + 3z^{-1} + 1)$$

$$G(z) = \frac{Y(z)}{U(z)} = \frac{2z^{-2} + 3z^{-1} + 1}{z^{-2} + 3z^{-1} + 2}$$

Transfer function generally:

$$\begin{aligned} y(k) + a_{n-1}y(k-1) + \dots + a_1y(k-n+1) + a_0y(k-n) &= \\ = b_nu(k) + b_{n-1}u(k-1) + \dots + b_1u(k-n+1) + b_0u(k-n) \end{aligned}$$

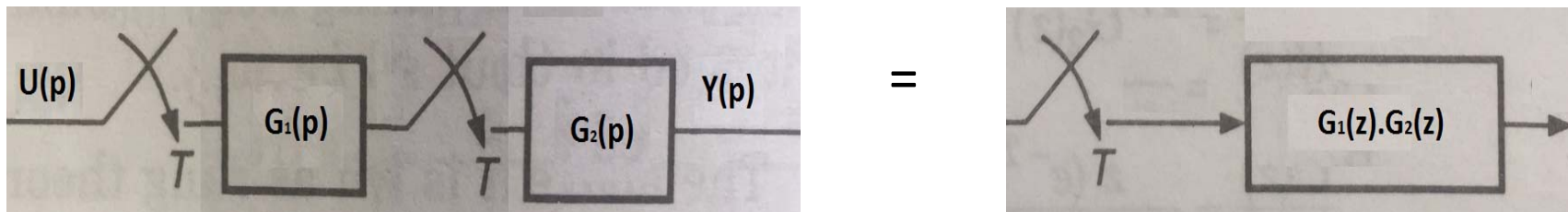
$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_mz^{-m} + b_{m-1}z^{-(m-1)} + \dots + b_0}{a_nz^{-n} + a_{n-1}z^{-(n-1)} + \dots + a_0}$$

Or

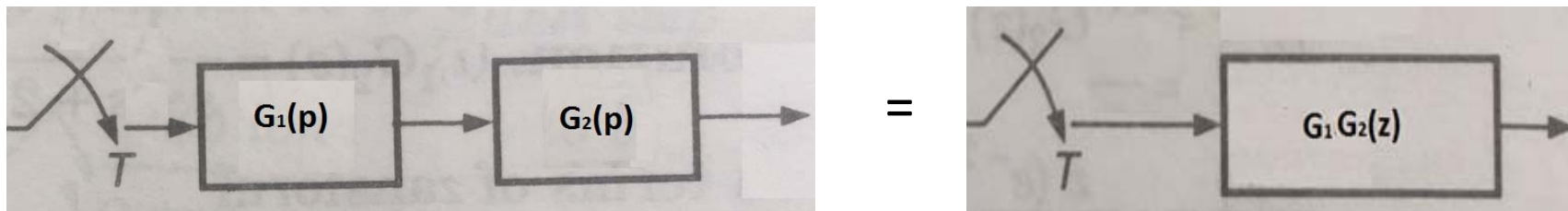
$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1z + \dots + b_mz^m}{a_0 + a_1z + \dots + a_nz^n}$$

Block diagram of transfer functions

a) Each block is sampled



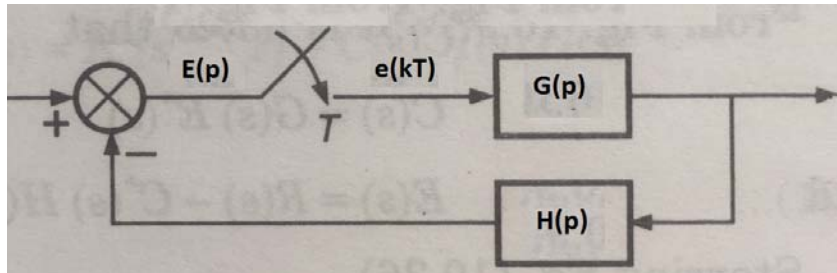
a) Sampler in the input of the first block



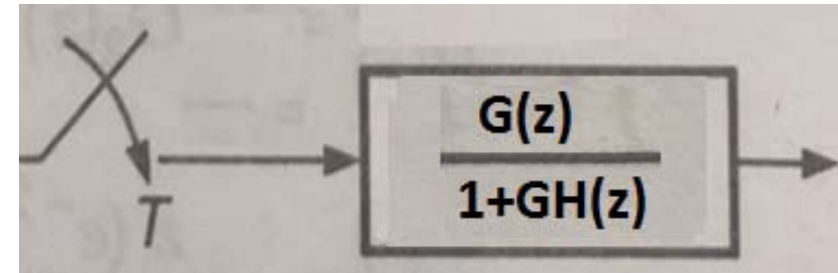
Block diagram of transfer functions

Closed-loop control system:

Error sampling



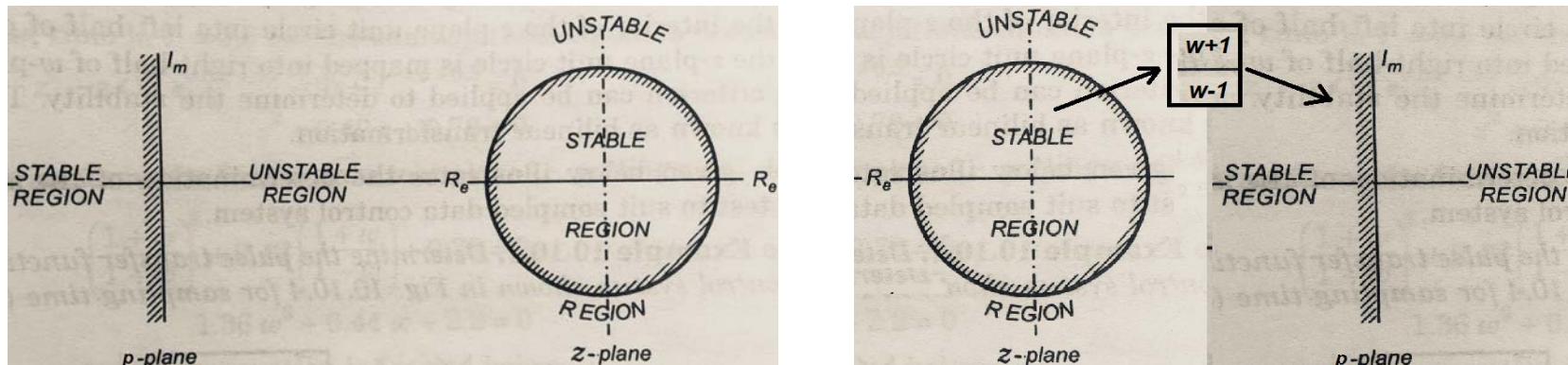
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Stability analysis of sampled-data control system

The stability of a sampled-data control system is determined by the location of the roots of characteristic equation (poles of overall transfer function).

The stability region in z-plane corresponding to p -plane is located by mapping from p -plane to z-plane.



If it is no simple to determine the roots, they can be identified by changing variable from z to w . This change of variable maps the interior of the z -plane unit circle into right-half of complex w -plane, while the exterior of the z -plane unit circle is mapped into left-half w -plane.

This mapping is known as **bilinear transformation**.
Therefore the Hurwitz criterion can be applied.

$$z = \frac{w + 1}{w - 1}$$

Bilinear transformation $z = \frac{w + 1}{w - 1}$

Illustrative example: Determine the stability of the control system given by transfer function.

$$G(z) = \frac{z^2 + 0,2z - 0,5}{z^3 - 1,2z^2 + 0,45z - 0,05}$$

Characteristic equation:

$$A(z) = z^3 - 1,2z^2 + 0,45z - 0,05 = 0$$

Bilinear transformation:

$$\left(\frac{w + 1}{w - 1}\right)^3 - 1,2\left(\frac{w + 1}{w - 1}\right)^2 + 0,45\left(\frac{w + 1}{w - 1}\right) - 0,05 = 0$$

$$(w + 1)^3 - 1,2(w + 1)^2(w - 1) + 0,45(w + 1)(w - 1)^2 - 0,05(w - 1)^3 = 0$$

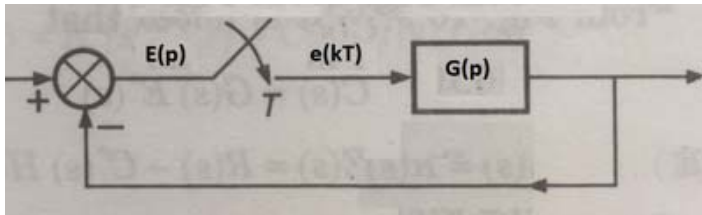
$$0,2w^3 + 1,5w^2 + 3,6w + 2,7 = 0$$

Hurwitz criterion:

$$H = \begin{vmatrix} 1,5 & 2,7 \\ 0,2 & 3,6 \end{vmatrix} = 4,86 > 0$$

Solved example

Determine the pulse transfer function and stability of the control system.



$$G_s(p) = \frac{5}{p(p + 0,5)}$$

$$G_s(p) = \frac{10}{p} + \frac{10}{p + 0,5}$$

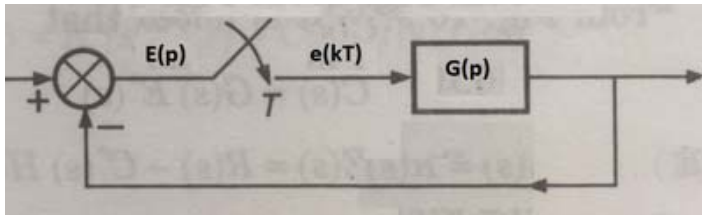
$$G_s(z) = 10 \left[\frac{z}{z-1} - \frac{z}{(z - e^{-0,5T})} \right] = \frac{10z(1 - e^{-0,5T})}{z^2 - z(1 - e^{-0,5T}) + e^{-0,5T}}$$

$$G_w(z) = \frac{G_s(z)}{1 + G_s(z)} = \frac{\frac{10z(1 - e^{-0,5T})}{z^2 - z(1 - e^{-0,5T}) + e^{-0,5T}}}{1 + \frac{10z(1 - e^{-0,5T})}{z^2 - z(1 - e^{-0,5T}) + e^{-0,5T}}} = \frac{10z(1 - e^{-0,5T})}{z^2 - z(11e^{-0,5T} - 9) + e^{-0,5T}}$$

$$A(z) = z^2 - z(11e^{-0,5T} - 9) + e^{-0,5T}$$

Solved example

Determine the pulse transfer function and stability of the control system.



$$G_s(p) = \frac{5}{p(p + 0,5)}$$

$$A(z) = z^2 - z(11e^{-0,5T} - 9) + e^{-0,5T}$$

For the sampling time **T = 0,5**:

$$z^2 + 0,42z + 0,78 = 0$$

$$z_{1,2} = -0,21 \pm 0,85i \quad \rightarrow \quad \text{System is stable, all the roots lie within unit circle of z-plane.}$$

For the sampling time **T = 1,0**:

$$z^2 - 2,33z + 0,606 = 0$$

$$z_1 = +2,10317$$

$$z_2 = +0,2983 \quad \rightarrow \quad \text{System is unstable, one root is located outside of unit circle.}$$

Conclusion: the sampling time can influence the stability of the system.

Discrete impulse and step response

- **Discrete impulse response** - is inverse Z-transform of the output $g(kT)$, if the input is specified as unit impulse (Dirac impulse) $\delta(k) = 1$ for $k=0$:

$$u(k) = \delta(k) = 1 \text{ for } k=0$$

$$U(z) = 1$$

$$Y(z) = G(z) U(z)$$

$$Y(z) = G(z)$$

- **Discrete step response** – is inverse Z-transform of the output $h(kT)$, if the input is specified as discrete unit step:

$$U(k) = 1 \text{ for } k \geq 0$$

$$U(z) = \frac{z}{z-1}$$

$$H(z) = G(z) U(z)$$

$$H(z) = G(z) \cdot \frac{z}{z-1}$$

References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.

ISBN: 81-7409-107-6

[2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing

[3] Matlab&Simulink: R2015a

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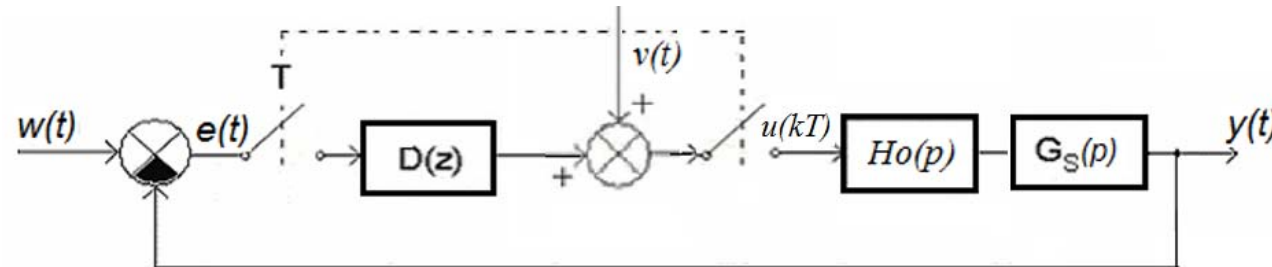


MINISTRY OF EDUCATION,
YOUTH AND SPORTS

14. Control Design in Discrete Time Domain

Sampled-data control system

Control system:



- $G_s(p)$ L-transfer function of controlled system
- $D(z)$ Z-transfer function of discrete controller (PC)
- $H_0(p)$ zero order hold
- $w(t)$ system input (or set point or reference), desired steady-state value of output
- $e(t)$ control deviation, error, difference between desired value and controlled value
- $u(t)$ actuating signal
- T sampling time of samplers
- $y(t)$ controlled value, output of control system

ZOH - zero-order hold

The original signal is reconstructed from the sampled signal by using a hold circuit. To use a hold circuit enables to hold the signal between two consecutive sampling instants.

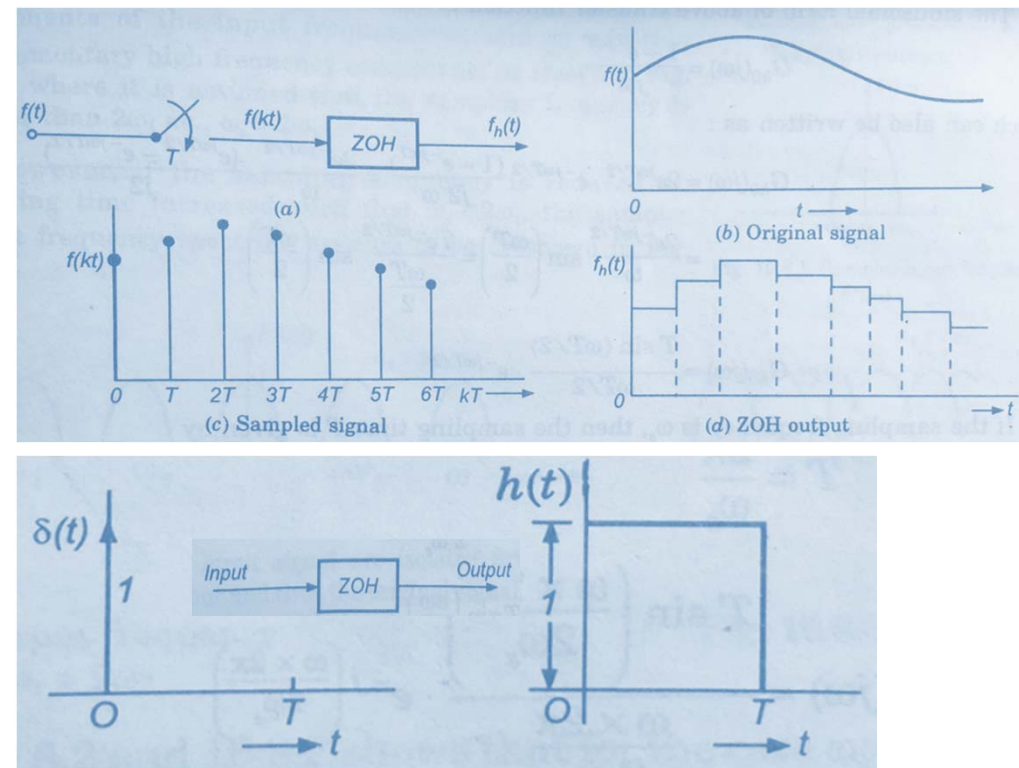
Zero-order hold holds the output signal at a fixed level between two consecutive sampling instants.

The output of ZOH:

$$h(t) = u(t) - u(t - T)$$

The transfer function of ZOH:

$$H_0(p) = \frac{1}{p} - \frac{1}{p} e^{-pT}$$



Total Z-transform of the continuous part of the control circuit

$$G_c(p) = H_0(p) G_s(p) = \frac{1 - e^{-Tp}}{p} G_s(p)$$

$$G_c(z) = Z \left\{ \frac{1 - e^{-Tp}}{p} G_s(p) \right\} = (1 - z^{-1}) Z \left\{ \frac{G_s(p)}{p} \right\} = \frac{P(z)}{Q(z)}$$

Illustrative example:

$$G_s = \frac{1}{p(p-a)}; \quad H_0(p) = \frac{1 - e^{-pT}}{p}$$

From Z-transform table:

$$G_c(z) = Z \left\{ \frac{1 - e^{-pT}}{p} \frac{1}{p(p+a)} \right\} = \frac{z-1}{z} \left[\frac{Tz}{(z-1)^2} - \frac{(1 - e^{-aT})z}{a(z-1)(z - e^{-aT})} \right]$$

T = sampling time

Example

Sampling time $T=5s$, consider zero order hold $H_o(p)$.

$$G_S(p) = \frac{1}{(5p+1)(p+1)}$$

Solution:

$$G_c(z) = Z \left\{ \frac{1-e^{-pT}}{p} \frac{1}{(5p+1)(p+1)} \right\} = \frac{0,5418z + 0,0861}{z^2 - 0,3746z + 0,0025}$$

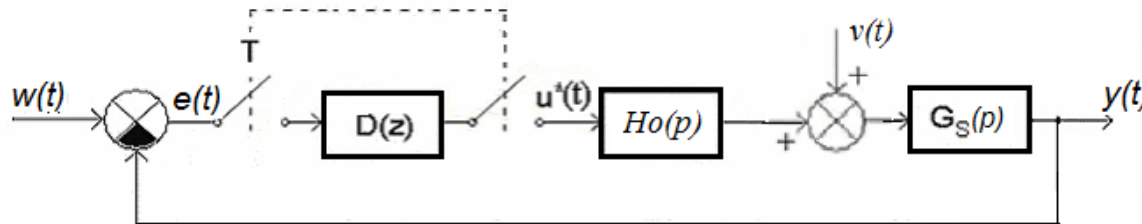
Solution in Matlab:

```
1 >> num=1
   num =
       1
   >> den=[5 6 1]
   den =
       5     6     1
```

```
2 >> [numd,dend]=c2dm(num,den,5,'zoh')
   numd =
       0    0.5418    0.0860
   dend =
       1.0000   -0.3746    0.0025
```

```
3 >> printsys(numd,dend,'z')
   num/den =
           0.5418  z  +   0.0860
   -----
          z^2  -   0.3746  z  + 0.0025
```

Synthesis of sampled data control system



1. According to the control transfer function requirements

$$G_w(z) = \frac{D(z) \cdot G_c(z)}{1 + D(z) \cdot G_c(z)} \Rightarrow D(z) = \frac{1}{G_c(z)} \cdot \frac{G_w(z)}{1 - G_w(z)}$$

2. According to the disturbance transfer function requirements

$$G_v(z) = \frac{G_c(z)}{1 + G_c(z)D(z)} \Rightarrow D(z) = \frac{G_c(z) - G_v(z)}{G_c(z)}$$

$D(z)$... transfer function of discrete regulator

$H_o(z)$... transfer function of zero-order hold

$G_c(z)$... total Z-transform of the continuous part of the control circuit

Control transfer function requirements

1. **The physical feasibility of the discrete regulator**
2. **Regulation to zero steady state deviation**
3. **The final duration of the transition process**
4. **Stability of the control circuit**

Control transfer function requirements

1. The physical feasibility of the discrete regulator:

$$G_c(z) = \frac{P(z)}{Q(z)} = g_m \cdot z^{-m} + g_{m+1} \cdot z^{-(m+1)} + \dots$$

$n \geq m$

$$G_w(z) = \frac{Y(z)}{W(z)} = f_n \cdot z^{-n} + f_{n+1} \cdot z^{-(n+1)} + \dots$$

$$D(z) = \frac{1}{g_m \cdot z^{-m} + g_{m+1} \cdot z^{-(m+1)} + \dots} \cdot \frac{f_n \cdot z^{-n} + f_{n+1} \cdot z^{-(n+1)} + \dots}{1 - (f_n \cdot z^{-n} + f_{n+1} \cdot z^{-(n+1)} + \dots)}$$

$$D(z) = \frac{1}{g_m \cdot z^{-m} + g_{m+1} \cdot z^{-(m+1)} + \dots} \cdot (h_n \cdot z^{-n} + h_{n+1} \cdot z^{-(n+1)} + \dots)$$

$$G_w(z) = f_1 \cdot z^{-1} + f_2 \cdot z^{-2} + \dots$$

Conclusion: the control transfer function must begin with a negative power of -1

Control transfer function requirements

2. Regulation to zero steady state deviation

$$\lim_{n \rightarrow \infty} e(nT) = \lim_{z \rightarrow 1} (1 - z^{-1}) \cdot E(z) = \lim_{z \rightarrow 1} (1 - z^{-1}) [W(z) - Y(z)] = 0$$

$$Y(z) = G_w(z) \cdot W(z)$$

$$\lim_{z \rightarrow 1} (1 - z^{-1}) \cdot W(z) [1 - G_w(z)] = 0$$

For different types of the input signal:

a) $w(t) = 1$ (*konstant input*)

$$1 - G_w(z) = (1 - z^{-1}) \cdot G(z)$$

b) $w(t) = t$ (*linear increasing - ramp fcn*)

$$1 - G_w(z) = (1 - z^{-1})^2 \cdot G(z)$$

c) $w(t) = t^2$ (*parabolic input*)

$$1 - G_w(z) = (1 - z^{-1})^3 \cdot G(z)$$

If there are any others requirements:

$$G_w(z) = z^{-1}$$

$$G_w(z) = 2 z^{-1} - z^{-2}$$

$$G_w(z) = 3 z^{-1} - 3 z^{-2} + z^{-3}$$

Where $G(z)$ is the optional polynomial

$$G(z) = g_0 + g_1 z^{-1} + g_2 z^{-2} + \dots$$

If there are any others requirements:

$$G(z) = 1$$

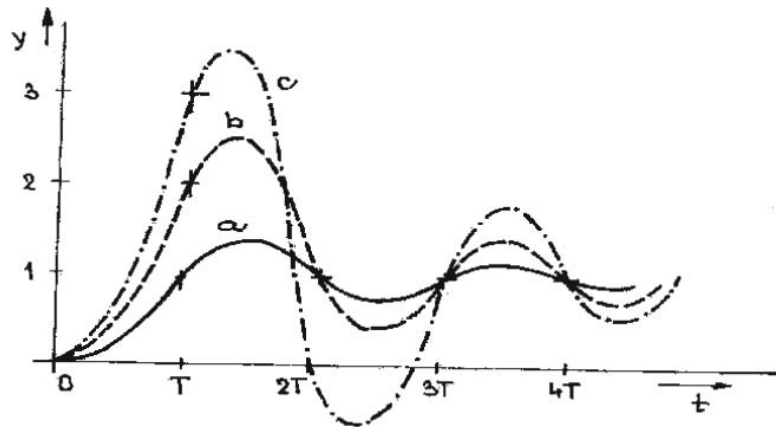
Control transfer function requirements

2. Regulation to zero steady state deviation

a) $G_W(z) = z^{-1}$ $Y(z) = z^{-1} + z^{-2} + z^{-3} + \dots$

b) $G_W(z) = 2z^{-1} - z^{-2}$ $Y(z) = 2z^{-1} + z^{-2} + z^{-3} + \dots$

c) $G_W(z) = 3z^{-1} - 3z^{-2} + z^{-3}$ $Y(z) = 3z^{-1} + z^{-3} + \dots$



Control transfer function requirements

3. The final duration of the transition process

The control transfer function must fulfilled this condition:

$$G_w(z) = P(z) \cdot R(z)$$

Where $P(z)$ is the numerator of the system transfer function

$R(z)$ is a optional polynomial: $R(z) = r_0 + r_1 z^{-1} + r_2 z^{-2} + \dots$

$$G_c(z) = \frac{P(z)}{Q(z)} \quad D(z) = \frac{A(z)}{B(z)}$$

$$G_w(z) = \frac{D(z) \cdot G_c(z)}{1 + D(z) \cdot G_c(z)} = \frac{A(z) \cdot P(z)}{B(z) \cdot Q(z) + A(z) \cdot P(z)}$$

Solved example

Transfer function of the system: $G_s(p) = \frac{1}{(5p+1)(p+1)}$ Input: $w(t) = 1$

Zero-order hold: $H_0(p) = \frac{1}{p} - \frac{1}{p} e^{-pT}$

Sampling time: $T = 5s$

We need zero steady state deviation and final transient time.

Solution:

Total Z-transform of the continuous part of the control circuit: $G_c(z) = \frac{0,5418z + 0,0861}{z^2 - 0,3746z + 0,0025}$

Request conditions:

1. Zero steady state deviation: $1 - G_w(z) = (1 - z^{-1}) \cdot G(z) = (1 - z^{-1})(1 + g_1 z^{-1})$

$$g_0 = 1$$

2. Final transient time: $G_w(z) = P(z) \cdot R(z) = (0,5418z + 0,0861) r_2 z^{-2}$

$$r_0 = 0$$

$$r_1 = 0$$

Solved example

Comparing coefficients of the both polynomials:

$$1 - (0,5418z + 0,0861) r_2 z^{-2} = (1 - z^{-1}) (1 + g_1 z^{-1}),$$

$$0,5418 \cdot r_2 = 1 - g_1 \quad r_2 = 1,5926$$

$$0,0861 \cdot r_2 = g_1 \quad g_1 = 0,1371$$

Required control transfer function: $G_w(z) = 0,8621 z^{-1} + 0,1371 z^{-2}$

$$\text{Discrete regulator: } D(z) = \frac{1}{G_c(z)} \cdot \frac{G_w(z)}{1 - G_w(z)} = \frac{1,5926 - 0,5965 z^{-1} + 0,0039 z^{-2}}{1 - 0,8629 z^{-1} - 0,1371 z^{-2}} = \frac{U(z)}{E(z)}$$

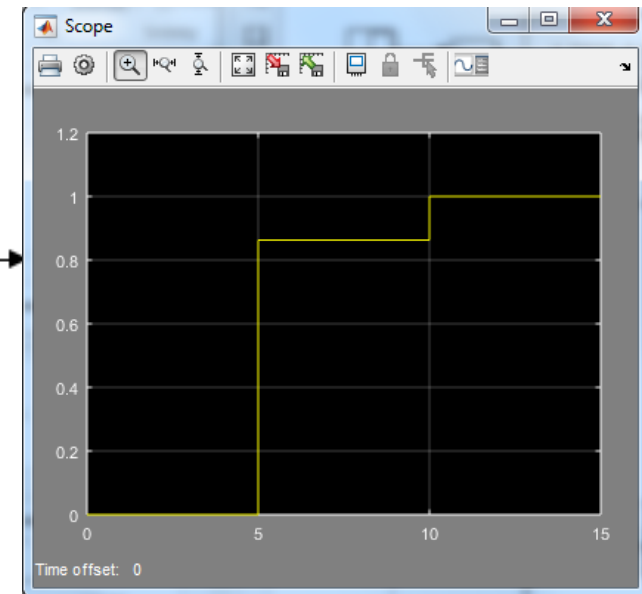
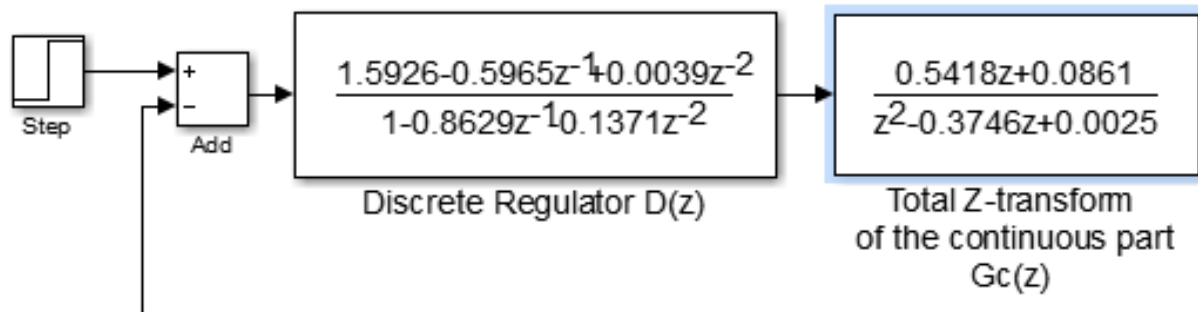
$$\begin{aligned} \text{Difference equation: } E(z) - 0,5965 E(z) z^{-1} + 0,0025 E(z) z^{-2} &= U(z) - 0,8629 U(z) z^{-1} - 0,1371 U(z) z^{-2} \\ e(k) - 0,5965 e(k-1) + 0,0025 e(k-2) &= u(k) - 0,8629 u(k-1) - 0,1371 u(k-2) \end{aligned}$$

Control algorithm for actuating signal:

$$u(k) = e(k) - 0,5965 e(k-1) + 0,0025 e(k-2) + 0,8629 u(k-1) + 0,1371 u(k-2)$$

Modelling in Matlab - Simulink

Block diagram and step response:



Control transfer function requirements

4. Stability of the control circuit

If: $G_c(z) = \frac{z+a}{z+b} \cdot Q_c(z)$ Where: $|a| \geq 1$ is unstable zero
 $|b| \geq 1$ is unstable pole

Then the polynomial of control transfer function must contain the terms:

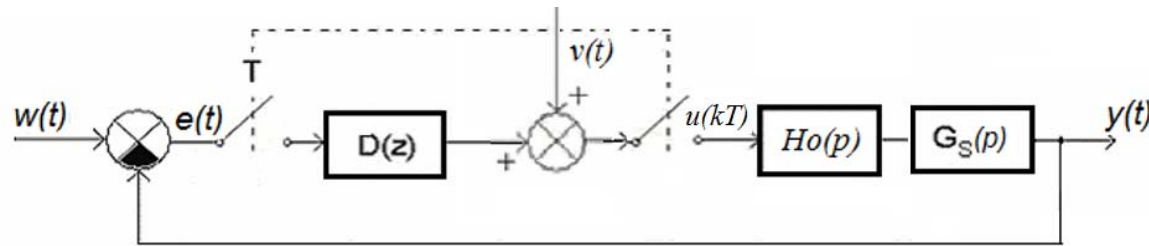
$$G_w(z) = (z+a) \cdot M(z)$$
$$1 - G_w(z) = (z+b)N(z)$$

Then the controller:

$$D(z) = \frac{1}{G_c(z)} \cdot \frac{G_w(z)}{1 - G_w(z)} = \frac{(z+b)}{(z+a)} \cdot \frac{1}{Q_c(z)} \cdot \frac{(z+a) \cdot M(z)}{(z+b) \cdot N(z)} = \frac{M(z)}{N(z) \cdot Q_c(z)}$$

Disturbance transfer function requirements

1. The physical feasibility of the discrete regulator



$$G_V(z) = \frac{G_c(z)}{1 + G_c(z)D(z)} = g_1 z^{-1} + g_2 z^{-2} + \dots$$

$$G_c(z) = \frac{P(z)}{Q(z)} = p_1 z^{-1} + p_2 z^{-2} + \dots$$

$$D(z) = \frac{G_c(z) - G_V(z)}{G_c(z)} = d_1 z^{-1} + d_2 z^{-2} + \dots \quad \Rightarrow$$

$$p_1 = g_1$$

Disturbance transfer function requirements

2. Regulation to zero steady state deviation

$$E(z) = -G_v(z)V(z) .$$

$$\lim_{n \rightarrow \infty} e(nT) = \lim_{z \rightarrow 1} (1 - z^{-1})E(z) = -\lim_{z \rightarrow 1} (1 - z^{-1})G_v(z)V(z) = 0$$

For different disturbance signal:

$v(t) = u_o$ (<i>konstant input</i>)	...	$G_v(z) = (1 - z^{-1}) \cdot G(z)$
$v(t) = at$ (<i>linear increasing - ramp fcn</i>)	...	$G_v(z) = (1 - z^{-1})^2 \cdot G(z)$
$v(t) = bt^2$ (<i>parabolic input</i>)	...	$G_v(z) = (1 - z^{-1})^3 \cdot G(z)$

Generally:

$$G_v(z) = (1 - z^{-1})^{k+1} \cdot G(z)$$

Where $G(z)$ is the optional polynomial: $G(z) = g_0 + g_1z^{-1} + g_2z^{-2} + \dots$

Disturbance transfer function requirements

3. The final duration of the transition process

$$G_v(z) = P(z) \cdot R(z)$$

3. Stability of the control circuit

$$G_v(z) = (z + a) \cdot M(z)$$

$|a| \geq 1$ Unstable zero

$$1 - G_v(z) = (z + b)N(z)$$

$|b| \geq 1$ Unstable pole

These conditions remain the same.

Solved example

Transfer function of the system: $G_s(p) = \frac{1}{(5p+1)(p+1)}$ Disturbance input: $v(t) = 1$

Zero-order hold: $H_0(p) = \frac{1}{p} - \frac{1}{p} e^{-pT}$

Sampling time: $T = 5s$

We need zero steady state deviation and final transient time.

Solution:

Total transfer function of the continuous part:

$$G_c(z) = Z \left\{ \frac{1 - e^{-pT}}{p} \frac{1}{(5p+1)(p+1)} \right\}_{T=5} = \frac{z-1}{z} Z \left\{ \frac{1}{p(5p+1)(p+1)} \right\}_{T=5} = \frac{0,5418z + 0,0861}{z^2 - 0,3746z + 0,0025} = \frac{P(z)}{Q(z)}$$

Using Matlab: `[NUMd, DENd] = C2DM(NUM, DEN, Ts, 'method')`

`[NUMd, DENd] = C2DM([1], [5 6 1], 5, 'zoh')`

Solved example

Requirements:

1. The physical feasibility of the discrete regulator: $g_1 = p_1 = 0,5418$
2. Regulation to zero steady state deviation: $G_v = (1 - z^{-1})(g_0 + g_1 \cdot z^{-1} + \dots) = (1 - z)(g_1 z^{-1} + g_2 z^{-2})$
3. The final duration of the transition process: $G_v = P(z) \cdot (r_0 + r_1 \cdot z^{-1} + \dots) = (0,5418 z^{-1} + 0,0861 z^{-2}) (1 + r_1 z^{-1})$

Comparing coefficients of the both polynomials:

$$g_1 = 0,5418$$

$$g_2 = 0,0861$$

$$r_1 = -1$$

Required disturbance transfer function: $G_v = 0,5418 \cdot z^{-1} - 0,4557 \cdot z^{-2} - 0,0861 \cdot z^{-3}$

Discrete regulator:

$$D(z) = \frac{1}{G_v(z)} - \frac{1}{G_c(z)} = \frac{1}{P(z)R(z)} - \frac{Q(z)}{P(z)} = \frac{1 - Q(z)R(z)}{P(z)R(z)} =$$

$$= \frac{1,3746 - 0,3771z^{-1} + 0,0025z^{-2}}{0,5418 - 0,4537z^{-1} - 0,0861z^{-2}} = \frac{U(z)}{E(z)}$$

Solved example

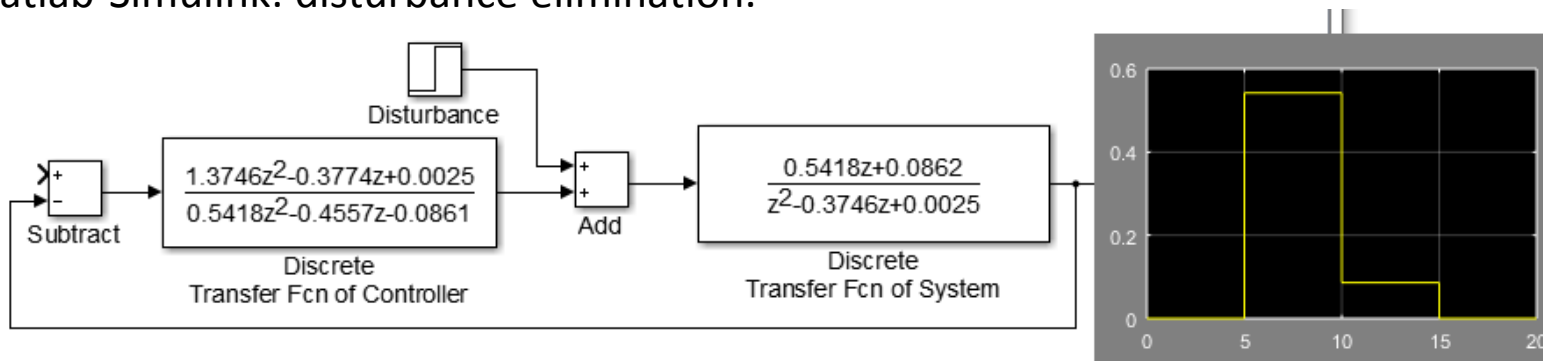
Difference equation: $1,3746E(z) - 0,3771E(z)z^{-1} + 0,0025E(z)z^{-2} = 0,5418U(z) - 0,4537U(z)z^{-1} - 0,0861U(z)z^{-2}$
 $1,3746e(k) - 0,3771e(k-1) + 0,0025e(k-2) = 0,5418u(k) - 0,4537u(k-1) - 0,0861u(k-2)$

Control algorithm for actuating signal:

$$u(k) = \frac{1}{0,5418} (1,3746e(k) - 0,3771e(k-1) + 0,0025e(k-2) + 0,4537u(k-1) + 0,0861u(k-2))$$

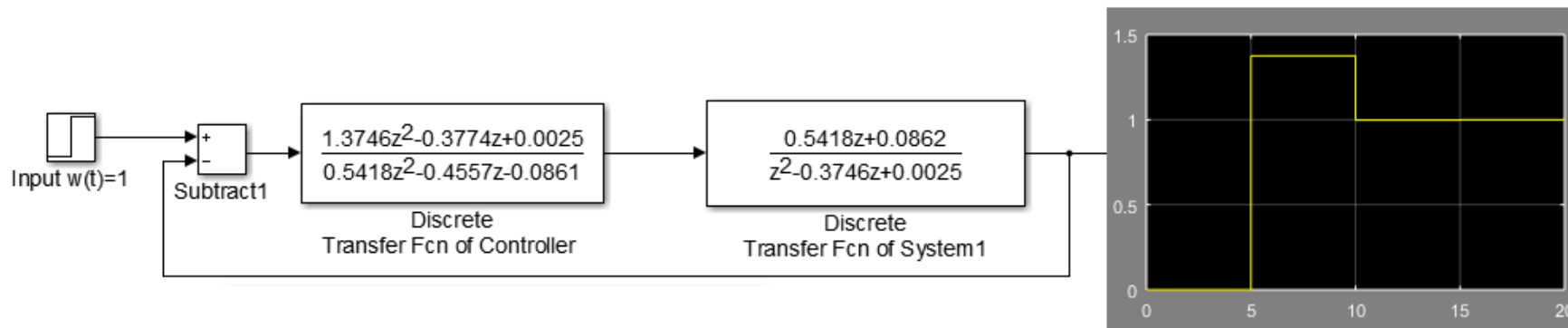
Control transfer function: $G_w(z) = \frac{D(z)G_c(z)}{1 + D(z)G_c(z)} = G_v(z)D(z) = P(z)R(z) \frac{1 - Q(z)R(z)}{P(z)R(z)} =$
 $= 1 - Q(z)R(z) = 1,3746 - 0,3771z^{-1} + 0,0025z^{-2}$

Simulation in Matlab-Simulink: disturbance elimination.



Solved example

Simulation in Matlab-Simulink: step response.



PSD - controllers

There are a three simple types of controllers according to the way they process a signal.

Type of controller	$u(t)$	$u(k)$	$U(z)$
P (P)	$r_0.e(t)$	$r_0.e(k)$	$r_0.E(z)$
S (I)	$r_{-1} \int_0^t e(\tau) d\tau$	$r_{-1} T \sum_{i=1}^k e(i)$	$r_{-1} T \frac{z}{z-1} E(z)$
D (D)	$r_d \frac{de(t)}{dt}$	$r_d \frac{e(k) - e(k-1)}{T}$	$\frac{r_d}{T} \frac{z-1}{z} E(z)$

Actuating signal in time domain: $u(k) = r_0.e(k) + r_{-1}T \sum_{i=1}^k e(i) + r_d \frac{e(k) - e(k-1)}{T}$

$$u(k) = K \left[e(k) + \frac{T}{T_i} \sum_{i=1}^k e(i) + \frac{T_d}{T} (e(k) - e(k-1)) \right]$$

Where: $r_0 = K$, $T_i = K/r_{-1}$, $T_d = r_d/K$

T ... sampling time

PSD - controllers

Actuating signal in Z-transform: $U(z) = K \left[E(z) + \frac{T}{T_i} \frac{z}{z-1} E(z) + \frac{T_d}{T} \frac{z-1}{z} E(z) \right]$

Discrete controller: $D(z) = \frac{U(z)}{E(z)} = K \left(1 + \frac{T}{T_i} \frac{z}{z-1} + \frac{T_d}{T} \frac{z-1}{z} \right) = \frac{K(z^2 - z + \frac{T}{T_i} z^2 + \frac{T_d}{T} (z-1)^2)}{z(z-1)}$

$$D(z) = \frac{K \left[(1 + \frac{T}{T_i} + \frac{T_d}{T}) z^2 - (2 \frac{T_d}{T} + 1) z + \frac{T_d}{T} \right]}{z^2 - z} = \frac{K(1 + \frac{T}{T_i} + \frac{T_d}{T}) - K(2 \frac{T_d}{T} + 1) z^{-1} + K \frac{T_d}{T} z^{-2}}{1 - z^{-1}} = \frac{d_0 - d_1 z^{-1} + d_2 z^{-2}}{1 - z^{-1}} = \frac{U(z)}{E(z)}$$

Actuating signal in time domain: $\Rightarrow u(n) - u(n-1) = d_0 e(n) - d_1 e(n-1) + d_2 e(n-2) \Rightarrow$

Where: $d_0 = K(1 + \frac{T}{T_i} + \frac{T_d}{T})$, $u(n) = u(n-1) + d_0 e(n) - d_1 e(n-1) + d_2 e(n-2)$

$$d_1 = K(1 + 2 \frac{T_d}{T})$$

$$d_2 = K \frac{T_d}{T}$$

PSD - controllers

PS – controller:
$$D(z) = \frac{d_0 - d_1 z^{-1}}{1 - z^{-1}}$$

Where:
$$d_0 = K(1 + \frac{T}{T_i})$$

$$d_1 = K$$

PD – controller:
$$D(z) = d_0 - d_1 z^{-1}$$

Where:
$$d_0 = K(1 + \frac{T_d}{T})$$

$$d_1 = K \frac{T_d}{T}$$

Solved example

For the system: $G_s = \frac{1}{(5p+1)(p+1)}$ was designed the continuous PI controller: $G_R = 0,5 \frac{5p+1}{p}$

Determine corresponding PS – controller $D(z) = \frac{d_0 - d_1 z^{-1}}{1 - z^{-1}}$

Solution:

Rearrange into standard PI form: $G_R = 0,5 \frac{5p+1}{p} = 2,5(1 + \frac{1}{5p})$ $K = 2,5$ $T_i = 5$

Discrete controller: $d_0 = K(1 - \frac{T}{T_i}) = 2,5(1 - \frac{1}{5}) = 3$

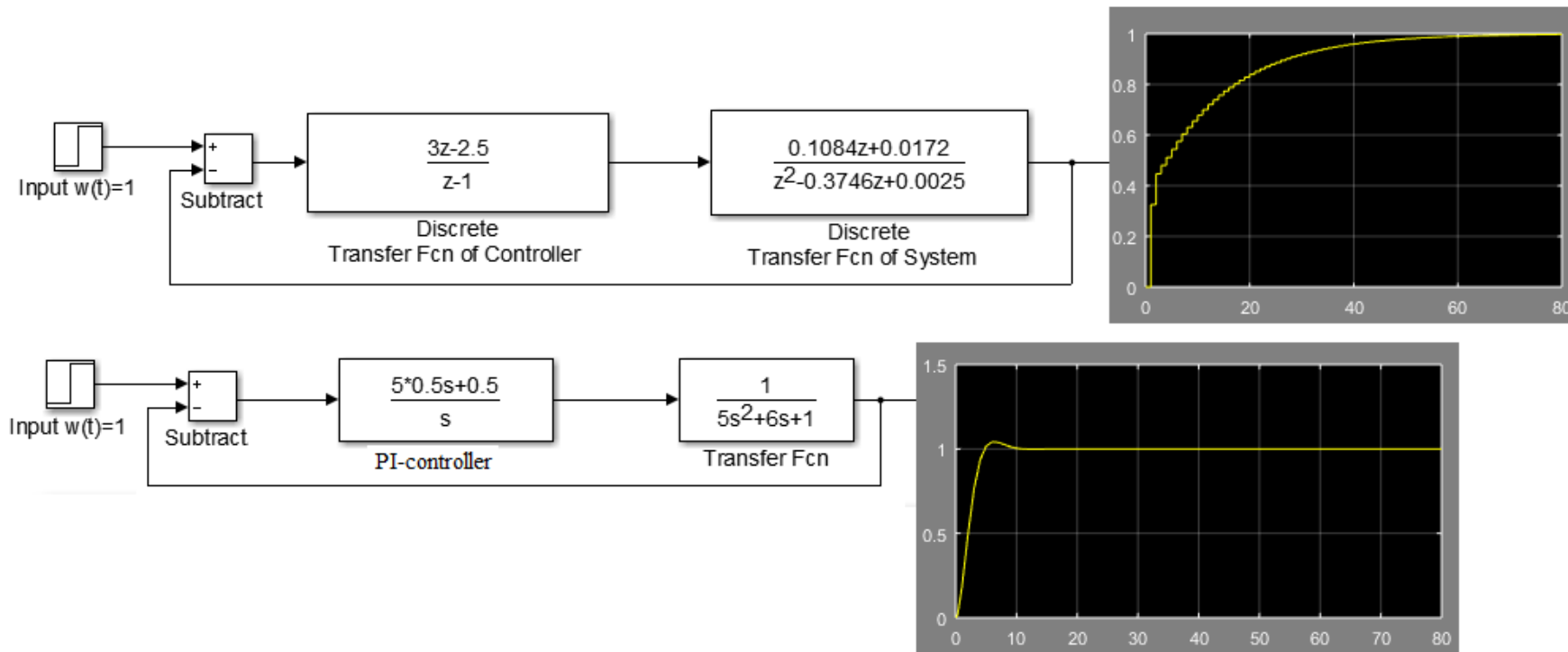
$$d_1 = K = 2,5$$

$$D(z) = \frac{3 - 2,5z^{-1}}{1 - z^{-1}} = \frac{U(z)}{E(z)}$$

Actuating signal: $u(n) = u(n-1) + 3e(n) - 2,5e(n-1)$

Solved example

Simulation in Matlab-Simulink:



References

[1] Manke, B., S.: Linear Control Systems with Matlab Applications, Khanna Publishers, 2009.

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[2] Chi-Tsong Chen: System and Signal Analysis, Saunders College Publishing

[3] Matlab&Simulink: R2015a

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